Trace Hardy Inequality for Euclidean Space with Cut

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Hardy Inequality

- The classical Hardy inequality in the Euclidean space $\mathbb{R}^d$, $d \geq 3$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \mathrm{d}x \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \mathrm{d}x, \quad \forall\, u \in H^1(\mathbb{R}^d)$$

- Various trace versions of the Hardy inequality, e.g.

$$\int_{\mathbb{R}^d_+} |\nabla u|^2 \mathrm{d}x \geq 2 \left(\frac{\Gamma(\frac{d}{4})}{\Gamma(\frac{d-2}{4})} \right)^2 \int_{\partial\mathbb{R}^d_+} \frac{|u|^2}{|x'|} \mathrm{d}x', \quad \forall\, u \in H^1(\mathbb{R}^d_+)$$

  where $\mathbb{R}^d_+$ denotes upper half-space in $\mathbb{R}^d$ for $d \geq 3$

- Usually holds for $d \geq 3$

Preliminaries

- We consider a connected, possibly unbounded, Lipschitz hypersurface $\Sigma \subset \mathbb{R}^d$ with two faces $\Sigma_\pm$
- The domain $\mathbb{R}^d \backslash \Sigma$ with a cut across Σ is connected
- The boundary $\partial\Sigma$ of Σ can be viewed as a $(d-2)$-dimensional manifold naturally embedded into $\mathbb{R}^d$.
- For any $x \in \Sigma$, we denote by $\rho_\Sigma(x)$ the geodesic distance between x and $\partial\Sigma$, measured in the induced Riemannian metric of Σ.
- We denote by $\mathcal{S}_{r_1,r_2}$ the open spherical shell centred at the origin with the inner radius $r_1 > 0$ and the outer radius $r_2 > r_1$.
- For any $u \in H^1(\mathbb{R}^d \backslash \Sigma)$, its traces $u|_{\Sigma_\pm}$ onto two faces $\Sigma_\pm$ of Σ are well-defined functions in $L^2(\Sigma)$.
- The jump of the trace $[u]_\Sigma := u|_{\Sigma_+} - u|_{\Sigma_-}$ is a well-defined and non-trivial function in $L^2(\Sigma)$.

Inequality for bounded cut

Theorem

Let $\Sigma \subset \mathbb{R}^d$ be a bounded Lipschitz hypersurface be as above. Then there exists a constant $C = C(\Sigma) > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathrm{d}x \geq C \int_{\Sigma} \frac{|[u]_\Sigma(x)|^2}{\rho_\Sigma(x)} \mathrm{d}\sigma(x), \quad \forall\, u \in H^1(\mathbb{R}^d \setminus \Sigma).$$

Sketch of proof

Lemma

Let $\Sigma \subset \mathbb{R}^d$ be a Lipschitz hypersurface as above. Then the following statements hold.

1. For any $\epsilon > 0$, there exists a constant $C_\epsilon = C_\epsilon(\Sigma) > 0$ such that

$$\|[u]_\Sigma\|^2_{L^2(\Sigma)} \leq \epsilon \|\nabla u\|^2_{L^2(\mathbb{R}^d;\mathbb{C}^d)} + C_\epsilon \|u\|^2_{L^2(\mathbb{R}^d)}$$

   holds for all $u \in H^1(\mathbb{R}^d \backslash \Sigma)$.

2. Assume that Σ is bounded and let $\Omega \subset \mathbb{R}^d$ be a bounded $\mathcal{C}^\infty$-smooth domain such that $\overline{\Sigma} \subset \Omega$. Then there exists a constant $\widetilde{C} = \widetilde{C}(\Omega, \Sigma) > 0$ such that

$$\int_\Sigma \frac{|[u]_\Sigma(x)|^2}{\rho_\Sigma(x)} \mathrm{d}\sigma(x) \leq \widetilde{C} \|u\|^2_{H^1(\Omega\backslash\Sigma)},$$

   holds for any $u \in H^1(\Omega \backslash \Sigma)$.

Sketch of proof

- Let $\Omega \subset \mathbb{R}^d$ be a bounded connected $\mathcal{C}^\infty$-smooth domain such that $\overline{\Sigma} \subset \Omega$ holds and $\Omega \backslash \overline{\Sigma}$ is connected.
- For any $u \in H^1(\mathbb{R}^d \setminus \Sigma)$ we have

$$\int_{\mathbb{R}^d} |\nabla_{\mathbb{R}^d \setminus \Sigma} u|^2 \mathrm{d}x \geq \int_{\Omega} |\nabla_{\Omega \setminus \Sigma} u_\Omega|^2 \mathrm{d}x$$

where $u_\Omega := u|_\Omega$

- The average of u_Ω is well defined

$$\langle u_\Omega \rangle = \frac{1}{|\Omega|} \int_\Omega u_\Omega(x) \mathrm{d}x$$

- It is easy to see that

$$[u_\Omega - \langle u_\Omega \rangle]_\Sigma = [u_\Omega]_\Sigma \quad \text{and} \quad \nabla_{\Omega \setminus \Sigma} \big(u_\Omega - \langle u_\Omega \rangle \big) = \nabla_{\Omega \setminus \Sigma} u_\Omega .$$

Sketch of proof

- The constant function on Ω is the eigenfunction corresponding to the lowest eigenvalue of the Neumann Laplacian on $\Omega\setminus\Sigma$
- The function $u_\Omega - \langle u_\Omega\rangle$ is orthogonal to constant function.
- By the min-max principle we can estimate the L^2-norm of the difference $\langle u_\Omega\rangle$ as

$$\lambda_2^{\mathrm{N}}(\Omega\setminus\Sigma)\|u_\Omega - \langle u_\Omega\rangle\|^2_{L^2(\Omega)} \leq \|\nabla_{\Omega\setminus\Sigma}(u_\Omega - \langle u_\Omega\rangle)\|^2_{L^2(\Omega;\mathbb{C}^d)}$$

  where $\lambda_2^{\mathrm{N}}(\Omega\setminus\Sigma) > 0$ denotes the second eigenvalue of the Neumann Laplacian
- we rewrite the inequality from Theorem as follows

$$\int_\Sigma \frac{|[u]_\Sigma(x)|^2}{\rho_\Sigma(x)}\mathrm{d}\sigma(x) = \int_\Sigma \frac{|[u_\Omega]_\Sigma(x)|^2}{\rho_\Sigma(x)}\mathrm{d}\sigma(x) = \int_\Sigma \frac{|[u_\Omega - \langle u_\Omega\rangle]_\Sigma(x)|^2}{\rho_\Sigma(x)}\mathrm{d}\sigma(x)$$
$$\leq \widetilde{C}\left(\|\nabla_{\Omega\setminus\Sigma}(u_\Omega - \langle u_\Omega\rangle)\|^2_{L^2(\Omega;\mathbb{C}^d)} + \|u_\Omega - \langle u_\Omega\rangle\|^2_{L^2(\Omega)}\right)$$
$$\leq \widetilde{C}(1 + (\lambda_2^{\mathrm{N}}(\Omega\setminus\Sigma))^{-1})\|\nabla_{\Omega\setminus\Sigma}(u_\Omega - \langle u_\Omega\rangle)\|^2_{L^2(\Omega;\mathbb{C}^d)}.$$

Conformal mapping

- Function M is smooth analytic complex function with non-zero derivative everywhere in $S \subset \mathbb{C}$
- conformal map acts as follows

$$\tilde{x} = \Re(M(x+iy))$$
$$\tilde{y} = \Im(M(x+iy))$$

- Cauchy-Riemann equations

$$\partial_x \tilde{x} = \partial_y \tilde{y} \quad \partial_x \tilde{y} = -\partial_y \tilde{x}$$

Linear fractional transformation-LFT

For $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ the mapping $M\colon \widehat{\mathbb{C}} \to \mathbb{C}$ is an LFT if one of the two conditions holds:
1) $c = 0,\ d \neq 0,\ M(\infty) := \infty$, and $M(z) := (a/d)z + (b/d)$ for $z \in \mathbb{C}$.
2) $c \neq 0,\ M(\infty) := a/c,\ M(-d/c) := \infty$, and $M(z) := \frac{az+b}{cz+d}$ for $z \in \mathbb{C}$, $z \neq -d/c$.

Conformal mapping

Lemma

Let $\mathcal{M}$ be an LFT as above with the Jacobian $\mathcal{J}_\mathcal{M}$. Then for any $x \in \mathbb{R}^2 \setminus \{\mathcal{Z}_\mathcal{M}\}$ and for any function $u \colon \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{C}$ differentiable at the point $\mathcal{M}(x)$ equality $|(\nabla v)(x)|^2 = |(\nabla u)(\mathcal{M}(x))|^2 \mathcal{J}_\mathcal{M}(x)$ holds with $v = u \circ \mathcal{M}$.

Lemma

Let $\Sigma \subset \mathbb{R}^2$ be an unbounded, piecewise-C^1 curve, parametrized via the unit-speed mapping $\sigma \colon (0,\infty) \to \mathbb{R}^2$. Let $\mathcal{M}$ be an LFT as above with the Jacobian $\mathcal{J}_\mathcal{M}$ and such that $\mathcal{Z}_\mathcal{M}, \mathcal{Z}_{\mathcal{M}^{-1}} \notin \overline{\Sigma}$. Let the bounded curve $\Lambda := \mathcal{M}^{-1}(\Sigma)$ be parametrized by $\lambda := \mathcal{M}^{-1} \circ \sigma$. For arbitrary $u \in H^1(\mathbb{R}^2 \setminus \Sigma)$ set $v := u \circ \mathcal{M}$. Then the following hold.

1. The identity $\int_{\mathbb{R}^2} |\nabla_{\mathbb{R}^2 \setminus \Sigma} u|^2 \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla_{\mathbb{R}^2 \setminus \Lambda} v|^2 \mathrm{d}x$ is satisfied.
2. If, in addition, $u \in C_c^\infty(\mathbb{R}^d \setminus \Sigma)$, then $\int_0^\infty [\mathcal{J}_\mathcal{M}(\sigma(s))]^{1/2} \left|[u]_\Sigma(s)\right|^2 \mathrm{d}s = \int_0^{|\Lambda|} \left|[v]_\Lambda(s)\right|^2 \mathrm{d}s$.

Conformal mapping approach

Theorem

Let an unbounded piecewise-C^1 curve $\Sigma \subset \mathbb{R}^2$ be parametrized by the mapping $\sigma\colon \mathbb{R}_+ \to \mathbb{R}^2$ with $|\dot\sigma(s)| \equiv 1$. Then there exists a constant $C = C(\Sigma) > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 \mathrm{d}x \geq C \int_0^\infty \frac{\big|[u]_\Sigma(s)\big|^2}{1 + |\sigma(s)|^2} \mathrm{d}s, \quad \forall\, u \in H^1(\mathbb{R}^2 \backslash \Sigma)$$

as long as there exists a conformal mapping M such that Σ is mapped to non-closed, bounded and non-intersecting curve.

Sketch of the proof: Application of previous theorems.

Inequality for unbounded cut

Admissible hypersurface

An unbounded Lipschitz hypersurface $\Sigma \subset \mathbb{R}^d$ is admissible if for an increasing unbounded sequence $0 = r_0 < r_1 < \dots r_n < \dots$, the domains $\mathcal{S}_{r_n, r_{n+1}} \setminus \Sigma$ are connected for all $n \in \mathbb{N}_0$.

Theorem

Let an unbounded Lipschitz hypersurface $\Sigma \subset \mathbb{R}^d$ be admissible. Then there exists a function $\mathcal{W}_\Sigma \colon \Sigma \to (0, \infty)$ such that

$$\int_{\mathbb{R}^d \setminus \Sigma} |\nabla u(x)|^2 \mathrm{d}x \geq \int_\Sigma \mathcal{W}_\Sigma(x) |[u(x)]_\Sigma(x)|^2 \mathrm{d}\sigma(x), \qquad \forall\, u \in H^1(\mathbb{R}^d \setminus \Sigma).$$

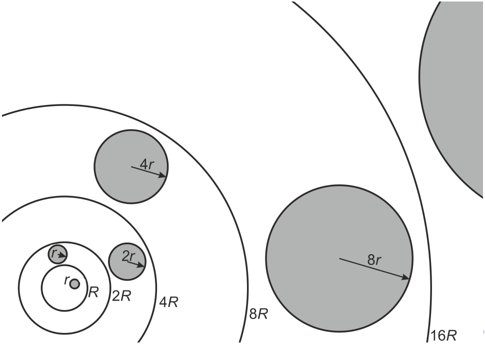
Sketch of the proof: Splitting of space into shells and working with them separately.

Perforated hyperplane

Perforated hyperplane

The surface $\Pi \subset \{(x', x_d) \in \mathbb{R}^d : x_d = 0\}$ is an admissible perforated hyperplane if for some $r > 0$ and $R > 2r$ holds

1. $\exists\, x_0 \in \mathcal{B}_r(0) \colon \Pi \cap \mathcal{B}_r(x_0) = \varnothing$.
2. $\exists\, x \in \mathcal{S}_{R+r,2R-r} \colon \Pi \cap \left(\cup_{n=1}^{\infty} \mathcal{B}_{2^{n-1}r}(2^{n-1}x) \right) = \varnothing$.



Perforated hyperplane

Theorem

Let $\Sigma \subset \mathbb{R}^d$ be an admissible perforated hyperplane. Then there exist $\hat{C} = \hat{C}(\Sigma) > 0$ and $\omega = \omega(\Sigma) > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathrm{d}x \geq \int_{\Sigma} \mathcal{W}(x) |[u]_{\Sigma}(x)|^2 \mathrm{d}\sigma(x), \qquad \forall\, u \in H^1(\mathbb{R}^d \setminus \Sigma),$$

with the weight

$$\mathcal{W}(x) := \begin{cases} \omega, & |x| < R \\ 2^{1-n}\omega, & 2^{n-1}R \leq |x| < 2^n R,\ n \in \mathbb{N}. \end{cases}$$

where R is the same as in Definition above.

Sketch of the proof: Splitting of space into shells and working with them separately.

Definition of the δ'-interaction on non-closed manifold

- the symmetric sesqulinear form

$$\mathfrak{a}^{\Gamma}_{\delta',\beta}[f,g] := (\nabla f_{\rm i}, \nabla g_{\rm i})_{\rm i} + (\nabla f_{\rm e}, \nabla g_{\rm e})_{\rm e} + \beta^{-1}(f_{\rm e}|_\Gamma - f_{\rm i}|_\Gamma, g_{\rm e}|_\Gamma - g_{\rm i}|_\Gamma)_\Gamma,$$
$$\mathrm{dom}\,\mathfrak{a}^{\Gamma}_{\delta',\beta} := \mathcal{H}^1(\Omega_{\rm e}) \oplus \mathcal{H}^1(\Omega_{\rm i}),$$

  is closed, densely defined and lower-semibounded in $L^2(\mathbb{R}^d)$
- linear mapping

$$\Lambda\colon \mathcal{H}^1(\Omega_{\rm e}) \oplus \mathcal{H}^1(\Omega_{\rm i}) \to L^2(\Gamma \setminus \Sigma), \qquad \Lambda f := f_{\rm e}|_{\Gamma\setminus\Sigma} - f_{\rm i}|_{\Gamma\setminus\Sigma},$$

- symmetric, densely defined and lower-semibounded form

$$\mathfrak{a}^{\Sigma}_{\delta',\beta}[f,g] := \mathfrak{a}^{\Gamma}_{\delta',\beta}[f,g], \qquad \mathrm{dom}\,\mathfrak{a}^{\Sigma}_{\delta',\beta} := \{f \in \mathrm{dom}\,\mathfrak{a}^{\Gamma}_{\delta',\beta} \colon \Lambda f = 0\}.$$

- the self-adjoint operator $-\Delta_{\Sigma,\beta}$ in $L^2(\mathbb{R}^d)$ is induced by the form $\mathfrak{a}^{\Sigma}_{\delta',\beta}$

Absence of negative eigenvalues for non-closed hypersurface

Key inequality

Let $\Gamma, \Sigma \subset \mathbb{R}^n$ be as above. Then there exists a constant $C = C(\Sigma) > 0$ such that

$$\|\psi_{\Sigma_+} - \psi_{\Sigma_-}\|^2_{L^2(\Sigma)} \leq C\|\nabla\psi\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^n)}$$

holds for any $\psi \in H^1(\mathbb{R}^n \setminus \Sigma)$.

Theorem

Let Λ be a non-closed compact Lipschitz manifold of the codimension 1. Then there exists β_0 such that the operator $-\Delta_{\Sigma,\beta}$ is positive as long as $\beta < \beta_0$. Furthermore the spectrum is $\sigma(-\Delta_{\Sigma,\beta}) = \sigma_{ess}(-\Delta_{\Sigma,\beta}) = \mathbb{R}^+$.

Summary

- Hardy inequality for a cut
- Holds also in dimension 2
- Works for bounded and unbounded cuts
- Application to δ'-interaction

Thank you for your attention