On the spectrum of a pair of particles on the half-line

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The model

- Two (distinguishable) particles moving on the half-line $\mathbb{R}_+ = (0, \infty)$.
- On the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ we consider the two-particle Hamiltonian

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + v\left(\frac{|x_1 - x_2|}{\sqrt{2}}\right)$$

The model

v a real-valued interaction potential such that:

- $v \in L^1_{loc}(\mathbb{R}_+)$ and $\max\{-v, 0\} \in L^\infty(\mathbb{R}_+)$
- The one-particle operator $h = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)$ on $L^2(\mathbb{R}_+)$ is such that $\inf \sigma(h) =: \varepsilon_0$ is an isolated (non-degenerate) eigenvalue
- $\varepsilon_0 < \liminf_{x\to\infty} v(x) := v_\infty$

Main results

Theorem

The essential spectrum of H is given by the interval $[\varepsilon_0, \infty)$. Furthermore, the discrete spectrum is non-empty and finite.

- Note that the two-particle Hamiltonian has ground state energy strictly lower than that of h.
- The existence of a discrete spectrum is a quantum geometrical effect. If one considers the pair on the full real line, no discrete spectrum exists.

Motivation

- The pairing of electrons plays a central role in the formation of the superconducting phase in metals (Cooper pairs).
- The discrete spectrum indeed leads to a Bose-Einstein condensation of non-interacting pairs of particles. Hence, geometrical effects lead to a Bose-Einstein condensation (see also Exner&Zagrebnov 2005, K. 2017/18)

Ideas on the proof: A reduction of the problem

- We actually prove the theorem for the operator Q_+, defined on $L^2(\Omega_0)$ where

$$\Omega_0 = \{(x_1, x_2) \in \mathbb{R}^2_+ : 0 < x_1 < x_2\}.$$

- $$Q_+[\varphi, \varphi] = \int_{\Omega_0} \left(|\nabla \varphi|^2 + v(x_1)|\varphi|^2 \right) \mathrm{d}x_1 \mathrm{d}x_2$$

  with form domain

$$\mathcal{D}(Q_+) = \{\varphi \in H^1(\Omega_0) : Q_+[\varphi, \varphi] < \infty\}$$

Ideas on the proof: Essential spectrum

In a first step one proves that $\inf \sigma_{ess}(H) \geq \varepsilon_0$:

- Here one employs an operator bracketing argument, dissecting Ω_0 into three domains using the two lines $x_1 = L$ and $x_2 = L$.
- Due to the lim inf-condition on the potential v, only the semi-infinite rectangle Ω_2 is important.
- By a separation of variables one concludes that $\inf \sigma_{ess}((-\Delta + v)|_{\Omega_2}) = \inf \sigma_{ess}(h_L^N)$, where h_L^N is the finite-volume version of h.
- The final step is to realise that $\inf \sigma_{ess}(h_L^N) \to \varepsilon_0$ as $L \to \infty$.

Ideas on the proof: Essential spectrum

In a second step one proves that $[\varepsilon_0, \infty) \subset \inf \sigma_{ess}(H)$:

- This is done by constructing a suitable Weyl sequence. We set, for any $k \in [0, \infty)$, $\varphi_n(x_1, x_2) := \psi_0(x_1)\tau(n - x_1) \cdot e^{ikx_2}\tau(x_2 - n)\tau(2n - x_2)$; here $\psi_0 \in D(h)$ is the ground state of h and $\tau : \mathbb{R} \to \mathbb{R}$ is a smooth function with $0 \leq \tau \leq 1$ and $\tau(x) = 1$ for $x \geq 2$ and $\tau(x) = 0$ for $x \leq 1$.
- A direct calculation then shows that

$$\frac{\|(-\Delta - (\varepsilon_0 + k^2))\varphi_n\|^2_{L^2(\Omega_0)}}{\|\varphi_n\|^2_{L^2(\Omega_0)}} \longrightarrow 0$$

as $n \to \infty$.

On the existence of a discrete spectrum

The general strategy is to find a function φ in the form domain of Q_+ such that

$$Q_+[\varphi, \varphi] - \varepsilon_0 \|\varphi\|^2_{L^2(\Omega_0)} < 0.$$

- We define $\varphi_n(x_1, x_2) := \psi_0(x_1)\phi_n(x_2)$
- Here $\phi_n(x_2) = \phi(x_2)\chi\left(\frac{x_2}{n}\right)$; χ is a smooth cut-off function with $0 \leq \chi \leq 1$ and $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$.
- Most importantly, for $\rho \in (1/2, 1)$, we set

$$\phi^{1/\rho}(x_2) := F(x_2) := \int_0^{x_2} |\psi_0(t)|^2 \mathrm{d}t.$$

On the existence of a discrete spectrum

- We note that φ_n is in the form domain of Q_+.
- A direct calculation then shows that $Q_+[\varphi_n, \varphi_n] - \varepsilon_0 \|\varphi_n\|^2_{L^2(\Omega_0)} < 0$ for n large enough.

On the finiteness of the discrete spectrum

The basic strategy is to reduce the two-dimensional problem to an effective one-dimensional one. This then allows one to employ well-known Bargmann estimates on the number of eigenvalue negative eigenvalues.

- For $R > 0$, we introduce the domains

$$\Omega_1 := \{(x_1, x_2) \in \Omega_0 : x_2 < x_1 + 2R\}$$

$$\Omega_2 := \{(x_1, x_2) \in \Omega_0 : x_2 > x_1 + R\}$$

and, $j = 1, 2$,

$$\chi_j^R(x_1, x_2) := \chi_j\left(\frac{x_2 - x_1}{R}\right)$$

with $\chi_1, \chi_2 : \mathbb{R} \to [0, \infty)$ such that $\chi_1(t) = 1$ for $t \leq 1$, $\chi_2(t) = 1$ for $t \geq 2$ as well as $\chi_1^2(t) + \chi_2^2(t) = 1$.

On the finiteness of the discrete spectrum

- A direct calculation shows that

$$Q_+[\varphi,\varphi] = Q_+[\chi_1^R\varphi,\chi_1^R\varphi] + Q_+[\chi_2^R\varphi,\chi_2^R\varphi] - \int_{\Omega_0} W_R|\varphi|^2\mathrm{d}x$$

with

$$W_R(x_1,x_2) := |\nabla\chi_1^R|^2 + |\nabla\chi_2^R|^2$$

- Consequently, we can introduce two operators Q_1, Q_2 on Ω_1, Ω_2 such that

$$Q_+[\varphi,\varphi] = Q_1[\chi_1^R\varphi,\chi_1^R\varphi] + Q_2[\chi_2^R\varphi,\chi_2^R\varphi].$$

- The operator Q_j differs from Q_+ on the corresponding domain Ω_j by adding the effective potential $-W_R$. Note that we impose Dirichlet boundary conditions along the defining lines of Ω_j.

On the finiteness of the discrete spectrum

- We denote by $N(A, \lambda)$ the number of eigenvalues (counted with multiplicity) below $\lambda \in \mathbb{R}$ of the self-adjoint operator A.
- From the previous relation we can compare Q_+ with $Q_1 \oplus Q_2$ (min-max principle) to obtain

$$N(Q_+, \varepsilon_0) \leq N(Q_1, \varepsilon_0) + N(Q_2, \varepsilon_0).$$

Hence, it remains to show that $N(Q_1, \varepsilon_0), N(Q_2, \varepsilon_0) < \infty$.

On the finiteness of the discrete spectrum

- To show that $N(Q_1, \varepsilon_0)$ is finite, one decomposes Ω_1 using the additional straight line $x_1 = L$. The $\liminf$-condition on v shows that the operator on the "outer" part (i.e., where $x_1 > L$) has no spectrum below ε_0. On the other hand, the remaining domain is bounded and hence the corresponding operator has purely discrete spectrum, implying the statement.

On the finiteness of the discrete spectrum

- Regarding $N(Q_2, \varepsilon_0)$ we introduce another comparison operator $\widehat{Q}_2$ for which one has $N(Q_2, \varepsilon_0) \leq N(\widehat{Q}_2, \varepsilon_0)$ (again by min-max principle).
- More explicitly, we define $\widehat{Q}_2$ on $L^2(\mathbb{R}_+ \times \mathbb{R})$ via its form

$$\widehat{Q}_2[\varphi, \varphi] := \int_{\mathbb{R}_+ \times \mathbb{R}} \left(|\nabla \varphi|^2 + (v(x_1) - W_R(x_1, x_2))|\varphi|^2 \right) \mathrm{d}x$$

$$D(\widehat{Q}_2) := \{\varphi \in H^1(\mathbb{R}_+ \times \mathbb{R}) : \widehat{Q}_2[\varphi, \varphi] < \infty\}.$$

- We introduce the projection Π via

$$(\Pi\varphi)(x_1, x_2) := \psi_0(x_1) \cdot \int_{\mathbb{R}_+} \overline{\varphi(x_1, x_2)} \psi_0(x_1) \mathrm{d}x_1$$

On the finiteness of the discrete spectrum

- A calculation then shows that

$$\widehat{Q}_2[\varphi,\varphi] \geq \widehat{Q}_2[\Pi\varphi,\Pi\varphi] - R\|W_R\Pi\varphi\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} + \left(E_2 - \frac{1}{R}\right)\|\Pi^\perp\varphi\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})} - W_R[\Pi^\perp\varphi,\Pi^\perp\varphi],$$

  where $E_2 := \inf\{\sigma(h)\setminus\varepsilon_0\}$.
- Hence, the first two terms define a self-adjoint operator A on $\mathrm{ran}\Pi$, and the last two terms a multiplication operator on $\mathrm{ran}\Pi^\perp$.
- Again by the min-max principle, we conclude that

$$N(\widehat{Q}_2,\varepsilon_0) \leq N(A,\varepsilon_0) + N(B,\varepsilon_0)$$

On the finiteness of the discrete spectrum

- One can show that, for sufficiently large $R > 0$, B has no spectrum in $(-\infty, \varepsilon_0)$ and hence $N(B, \varepsilon_0) = 0$.
- Finally, A is effectively a one-dimensional Schrödinger operator with some effective potential. Classical estimates (Bargmann estimates) then show that $N(A, \varepsilon_0) < \infty$.

On the finiteness of the discrete spectrum

- We remark that no bound on the number of eigenvalues in the discrete spectrum was derived!
- However, if one considers the initial case where the potential v was informally defined as, $d > 0$,

$$v(x) := \begin{cases} 0 \quad \text{for} \quad x < d/\sqrt{2} \,, \\ \infty \quad \text{else} \,, \end{cases}$$

then one can show that the discrete spectrum consists of exactly one eigenvalue.

Thank you for your attention!

S. Egger, J. Kerner, and K. Pankrashkin *Bound states of a pair of particles on the half-line with a general interaction potential*, arXiv:1812.06500.

J. Kerner *On the number of isolated eigenvalues of a pair of particles in a quantum wire*, arXiv:1812.11804.