Waveguides with asymptotically diverging twisting

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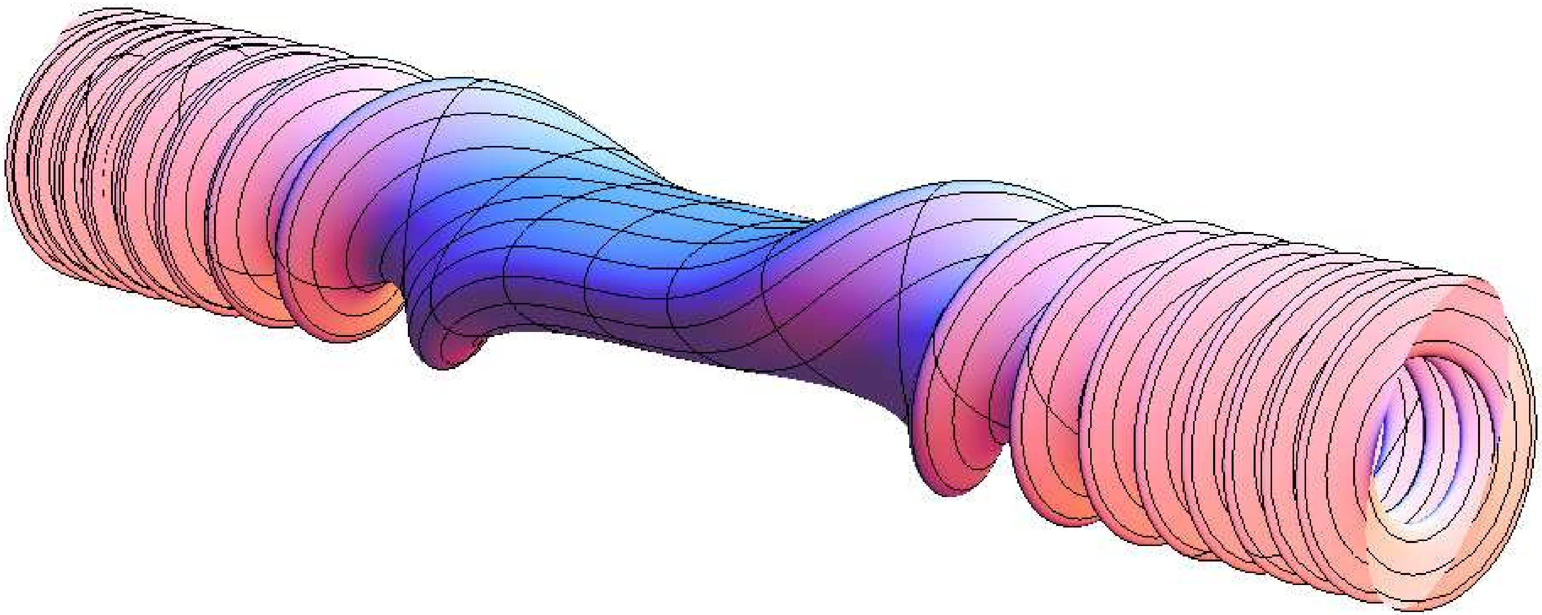

Based on:

- Applied Mathematics Letters 46 (2015) [with nobody]
- Annales Henri Poincaré 19 (2018) [with Rafael Tiedra de Aldecoa]
- ZAMP (2019) [with Philippe Briet & Hamza Abdou-Soimadou]

Outline

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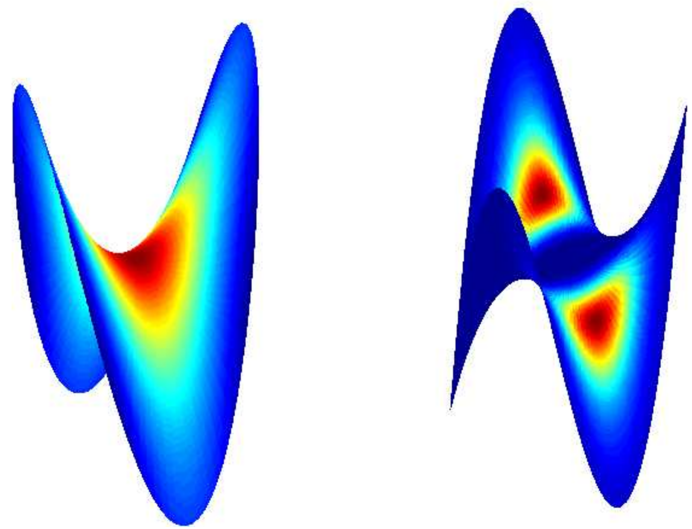
0. Spectral-geometric motivations



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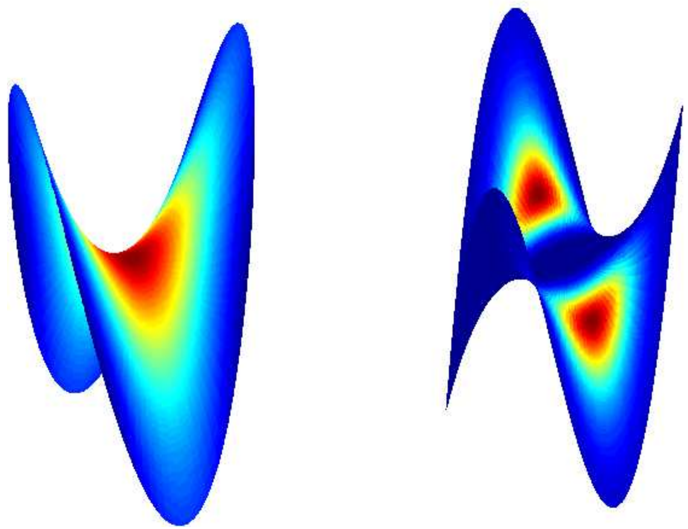
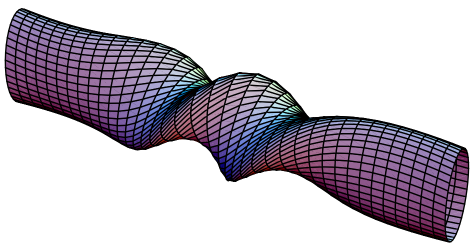
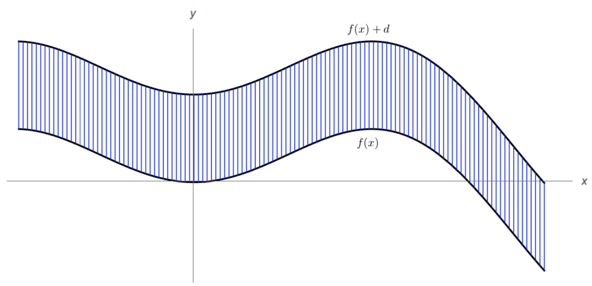


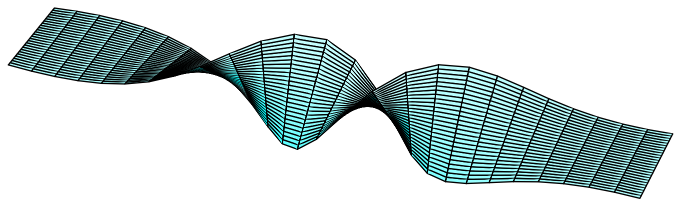
1. Twisted tubes



2. Sheared ribbons



3. Ruled strips



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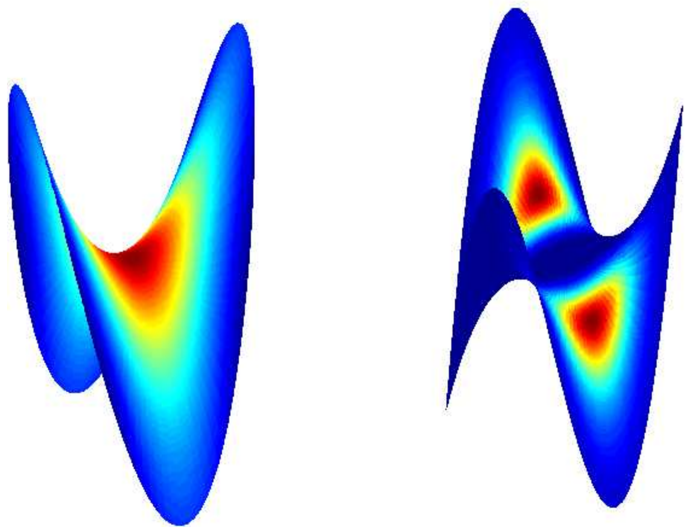

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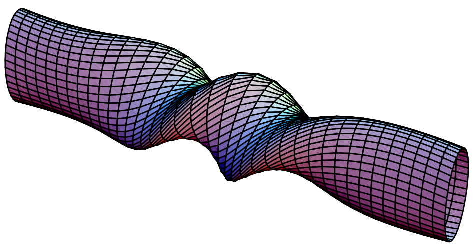

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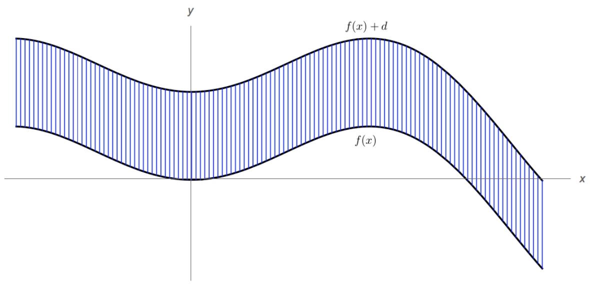

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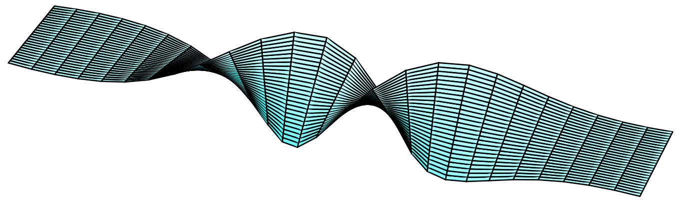
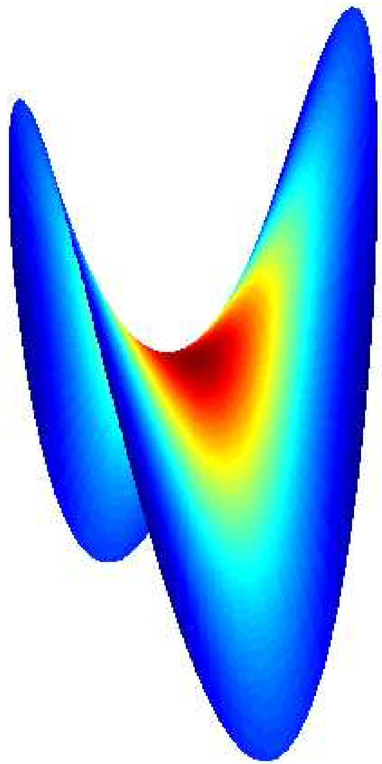
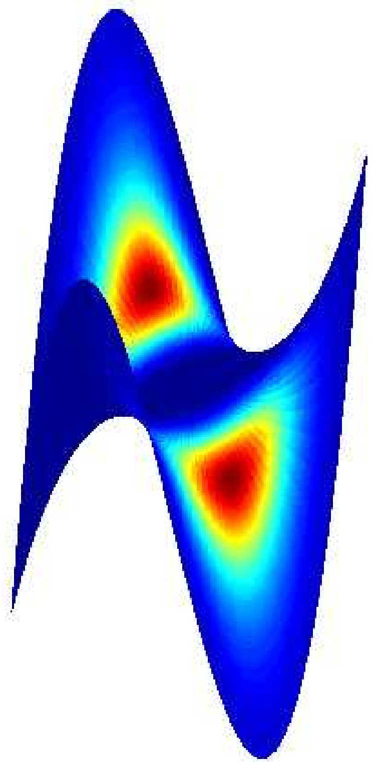


4. Conclusions

Part 0.

Spectral-geometric motivations





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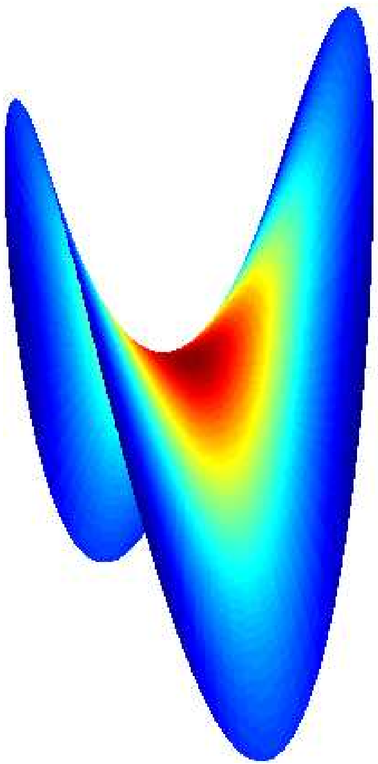
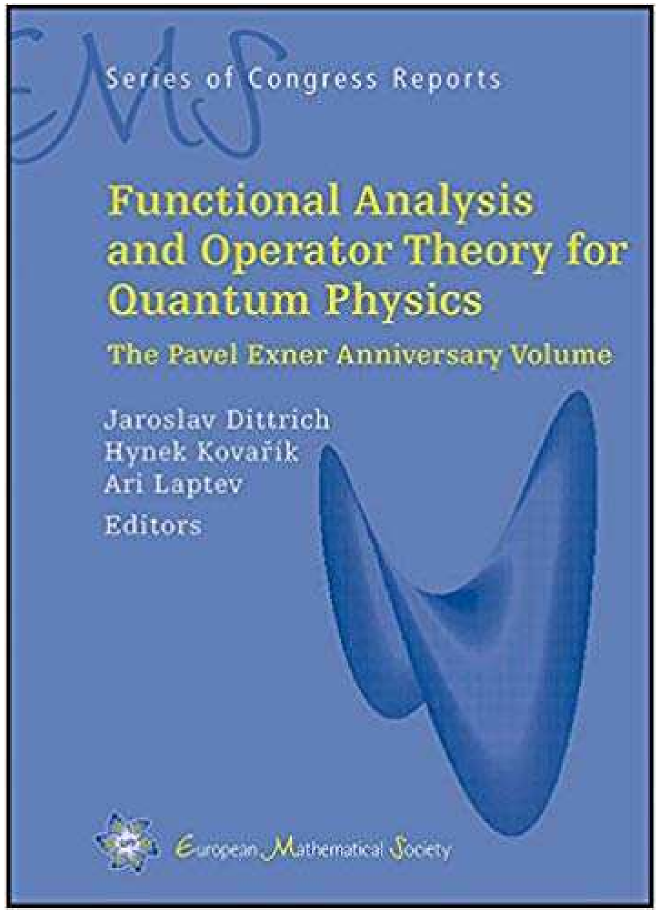
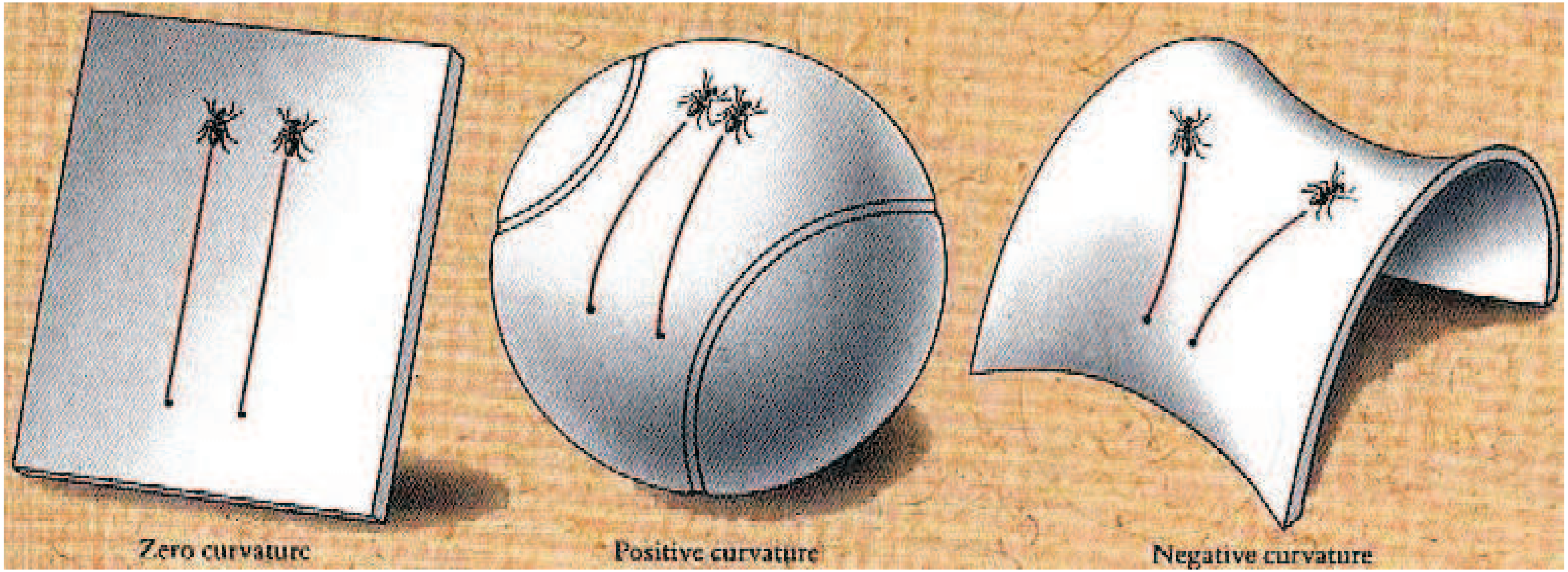




Which geometry is better to travel in ?

$K = 0$ $K > 0$ $K < 0$



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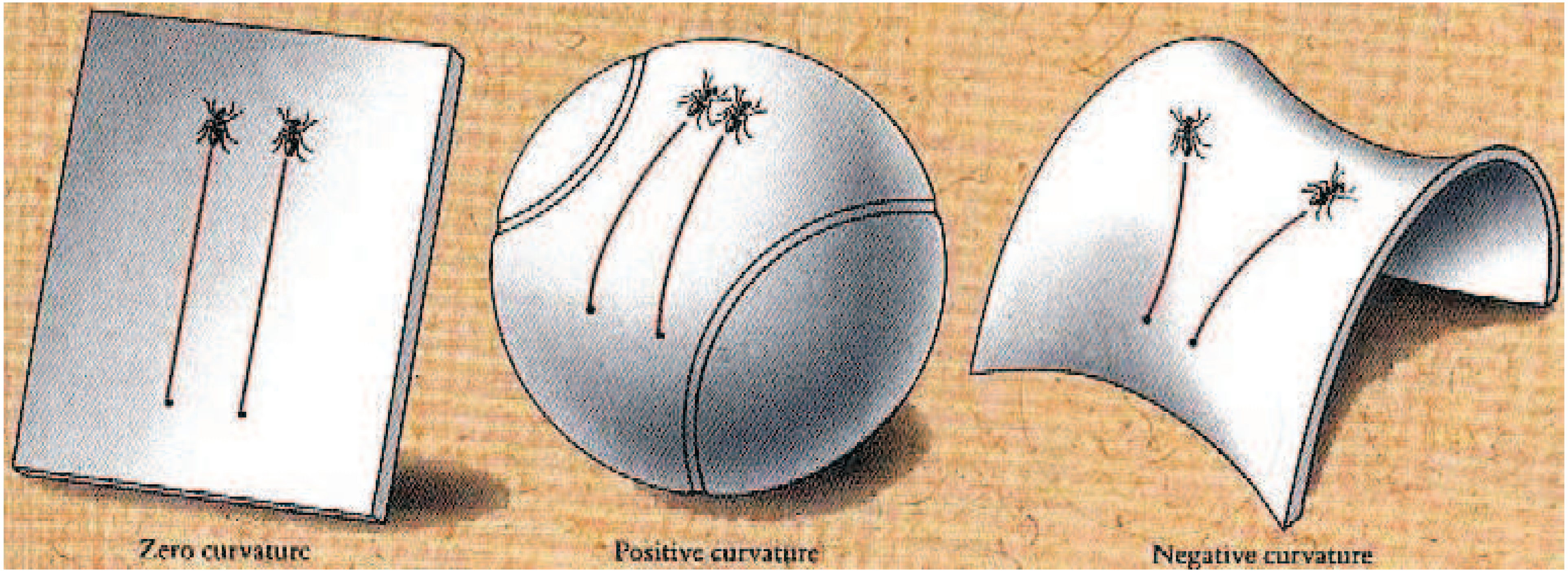

critical bad good

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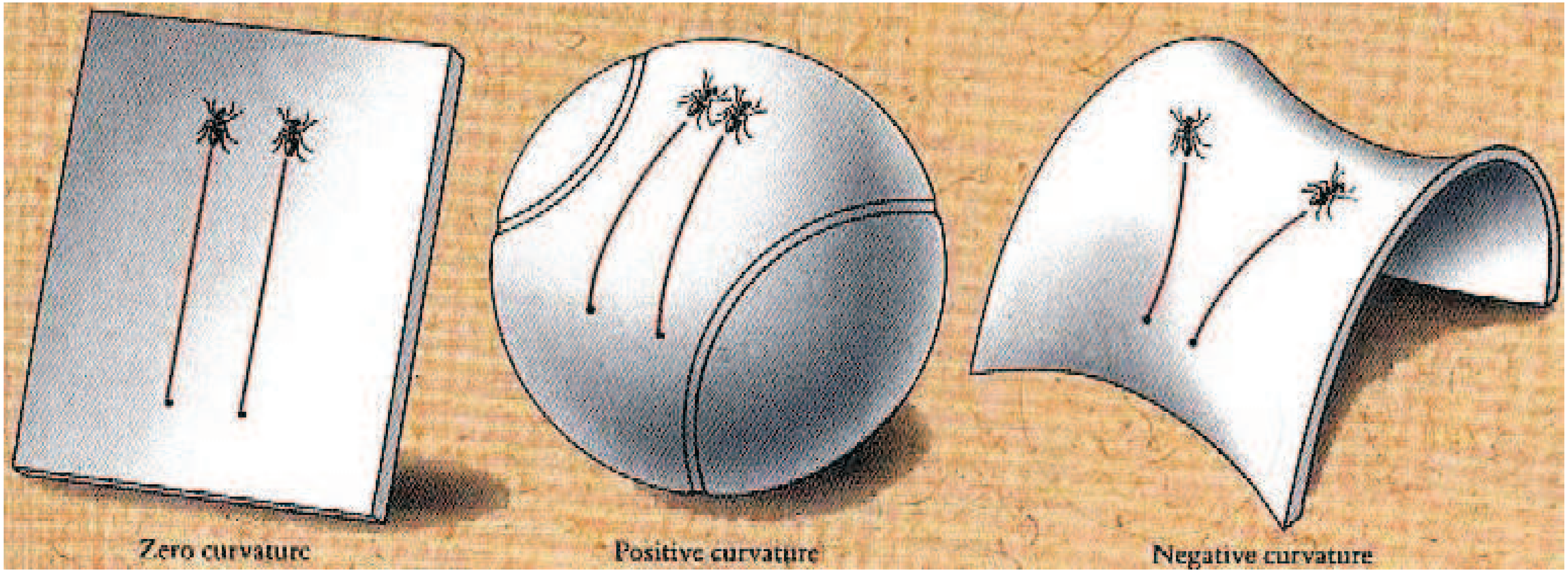

critical | bad | good

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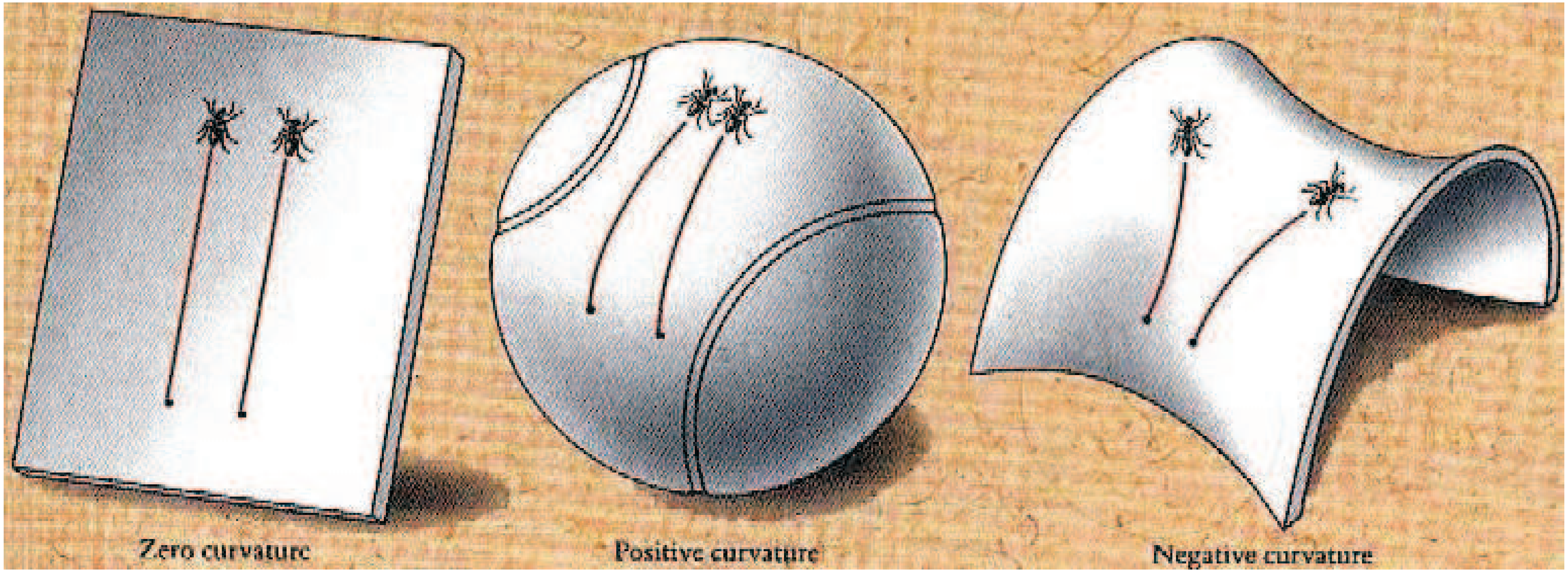

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$$\left\| e^{-t(-\Delta)} \right\| = e^{-t \min \sigma(-\Delta)}$$

From super-transport to no-transport

[Donnelly, Li 1979]

Let $-\Delta$ be the Laplacian of a complete surface with finitely generated fundamental group.

$$K(x) \xrightarrow[|x|\to\infty]{} -\infty$$

$$\Downarrow$$

$$\sigma(-\Delta) = \sigma_{\text{disc}}(-\Delta)$$

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$$B = \operatorname{rot} A$$

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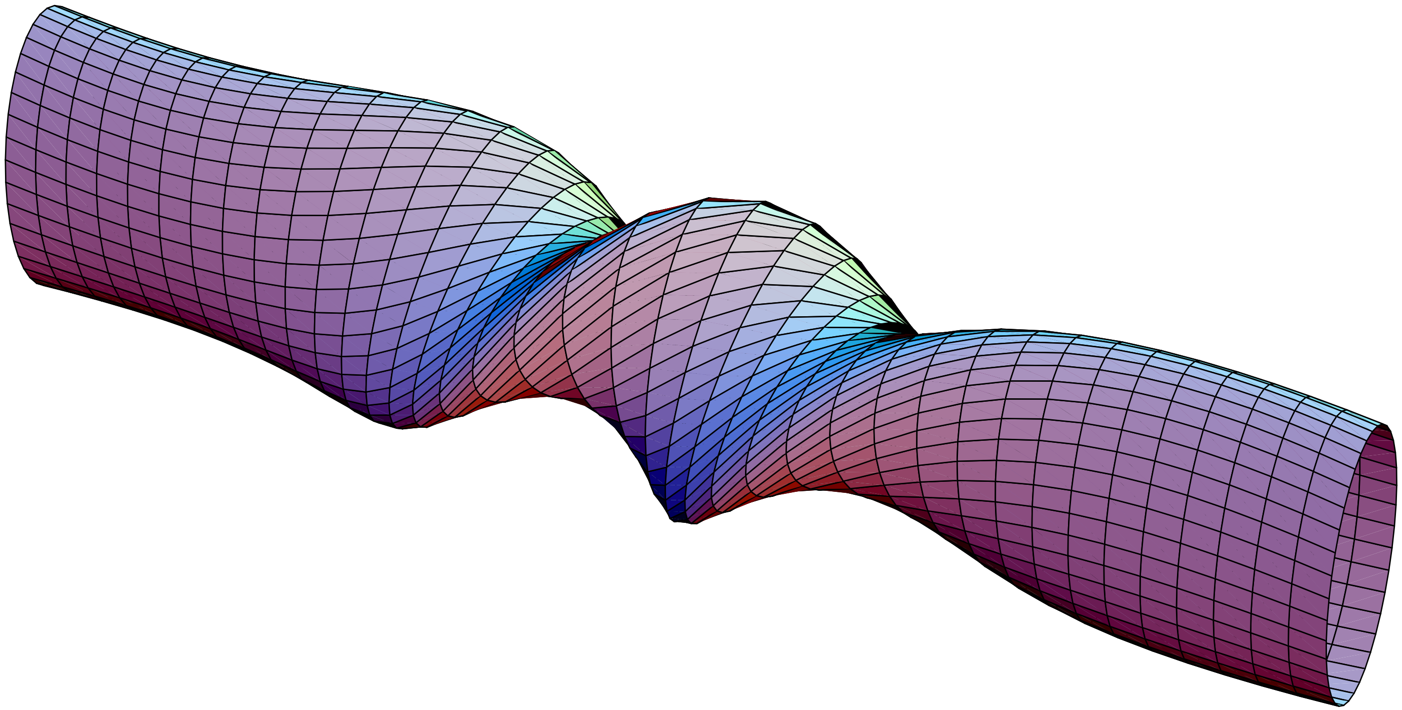
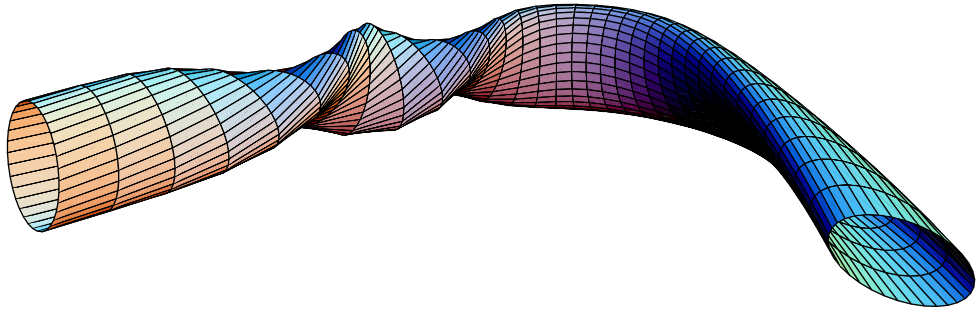


Twisting versus bending in curved waveguides

$-\Delta_D^\Omega$ in $L^2(\Omega)$



$\Omega :=$ *local* deformation of $\mathbb{R} \times \omega$
$\omega \subset \mathbb{R}^2$ bounded domain (cross-section)

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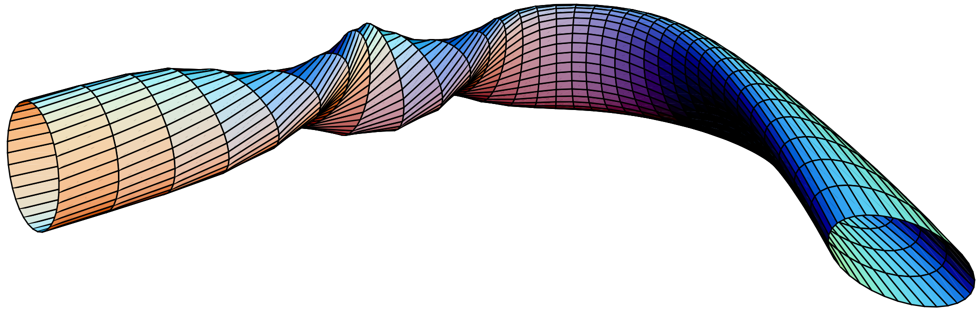

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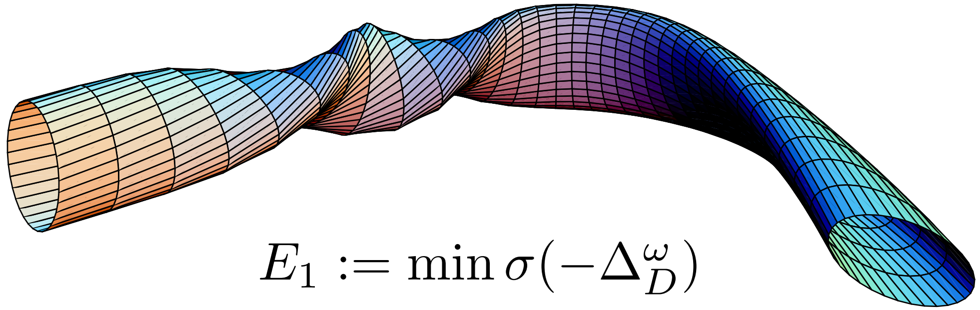

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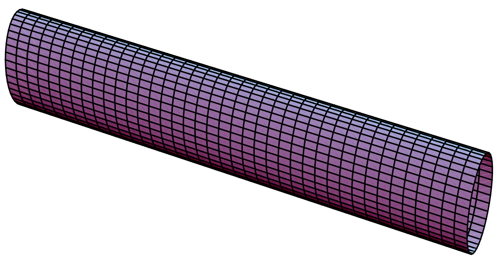
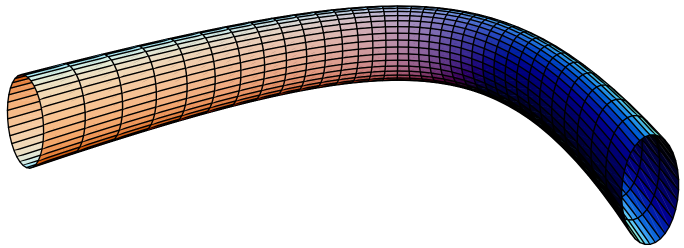
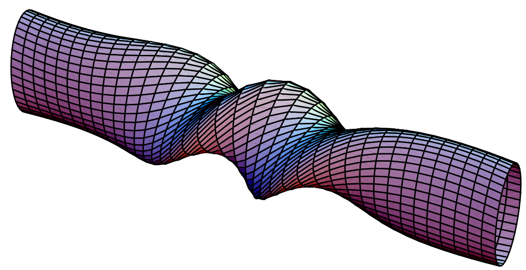
geometry of Ω $\longleftrightarrow$ spectrum of $-\Delta_D^\Omega$

Ω	straight (1800 ?)[1]	bent (1989)[2]	twisted (2008)[3]
interaction	*null*	*attractive*	*repulsive*
spectrum	0 E_1	0 E_1	0 E_1
math	criticality	bound states	Hardy inequalities







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[2] Exner - Šeba, . . .
[*J. Math. Phys.* 30]

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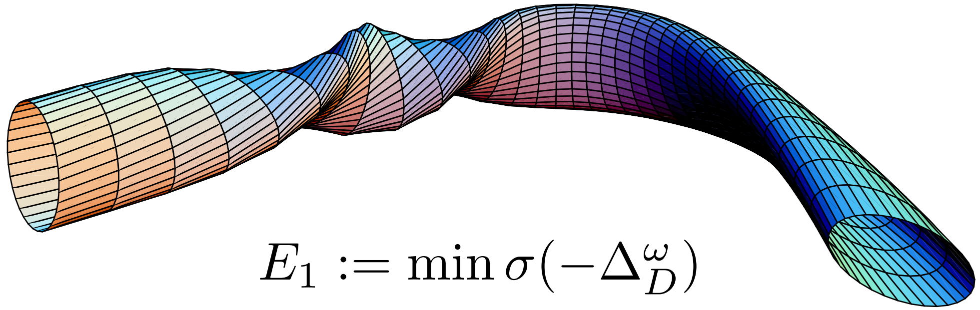

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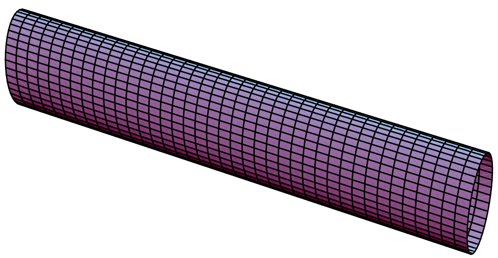

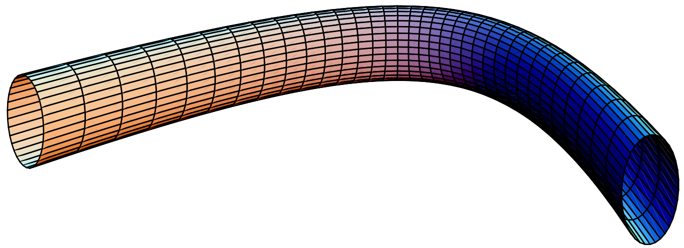

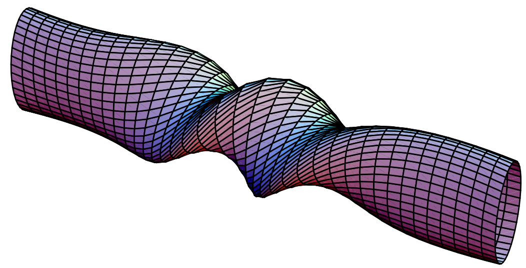

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From local to global twisting

$$\Omega := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} s \\ t_1 \\ t_2 \end{pmatrix} : (s, t_1, t_2) \in \mathbb{R} \times \omega \right\}$$

$\theta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$

twisting angle

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$$-\Delta_D^\Omega - E_1 \geq \frac{c}{1+s^2}$$

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[Briet, Kovařík, Raikov, Soccorsi 2009], [D.K., Zuazua 2010], [Briet, Hammedi, D.K. 2015], ...

From global to diverging twisting

$$|\theta'(s)| \xrightarrow[|s|\to\infty]{} \infty$$

[D.K. *Appl. Math. Lett.* 2015]

¡ $|\Omega| = \infty$!

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Theorem 1 (quasi-conical realisation). If $0 \in \omega$, then

$$\sigma(-\Delta_D^\Omega) \supset [\mu_1, \infty)$$

(essential spectrum is not empty)

where $\mu_1 := \min \sigma(-\Delta_D^{B_r})$ with $r := \mathrm{dist}(0, \partial\omega)$.

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Theorem 2 (quasi-bounded realisation). If $\omega \subset \{(t_1, t_2) \in \mathbb{R}^2 : \ t_1 > 0\}$, then

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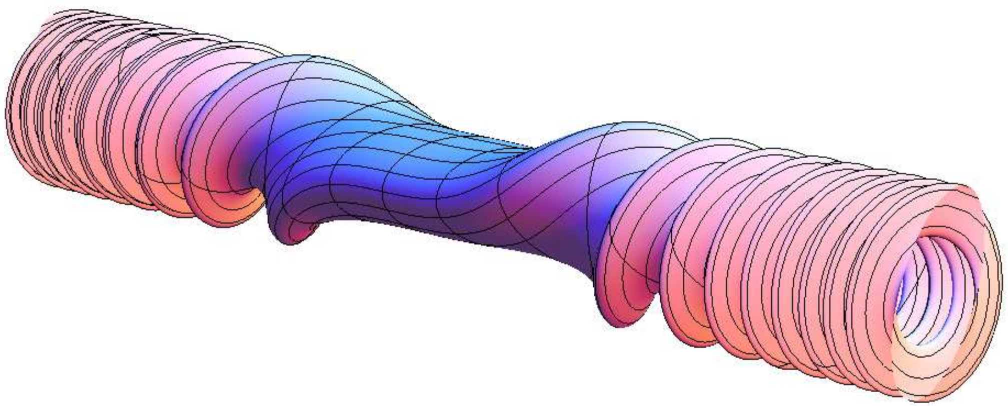
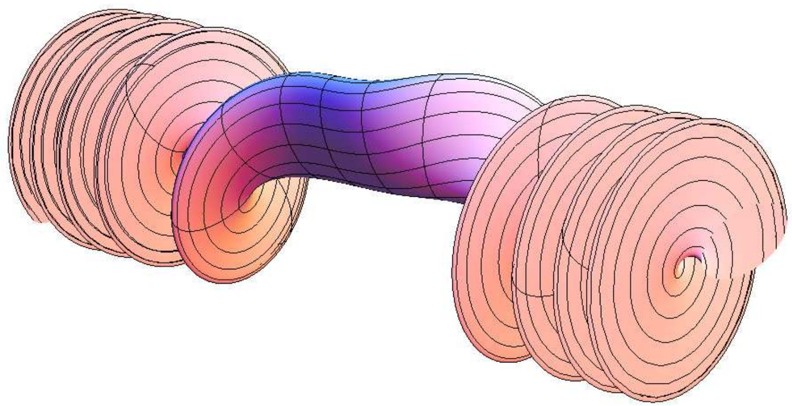
Proof.





q.e.d.

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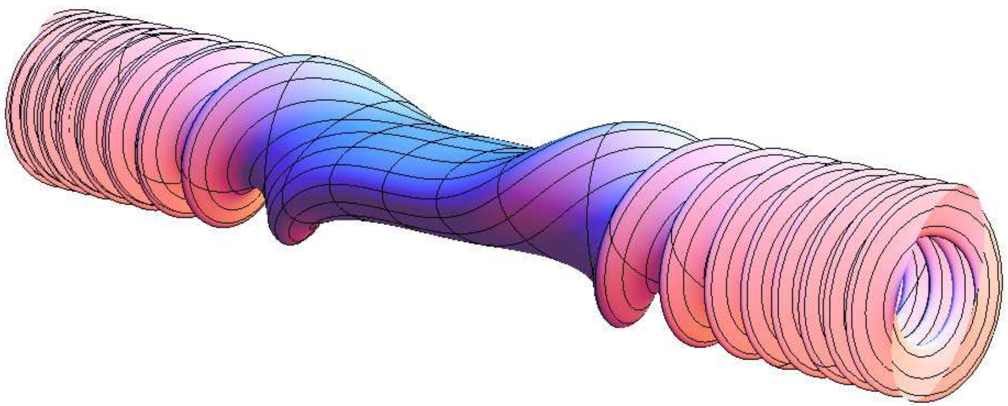
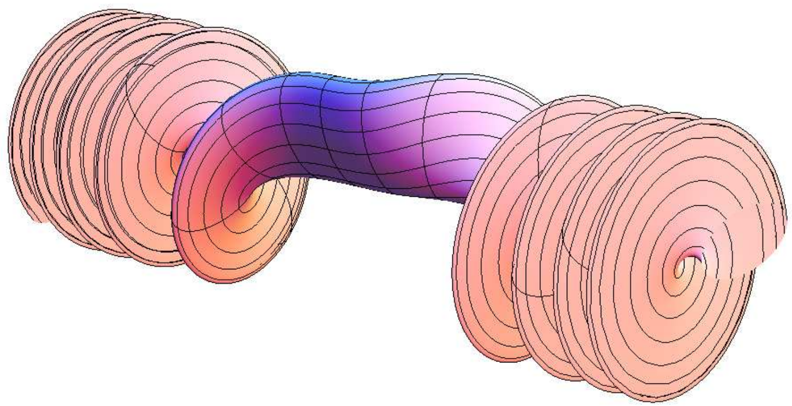
Proof.



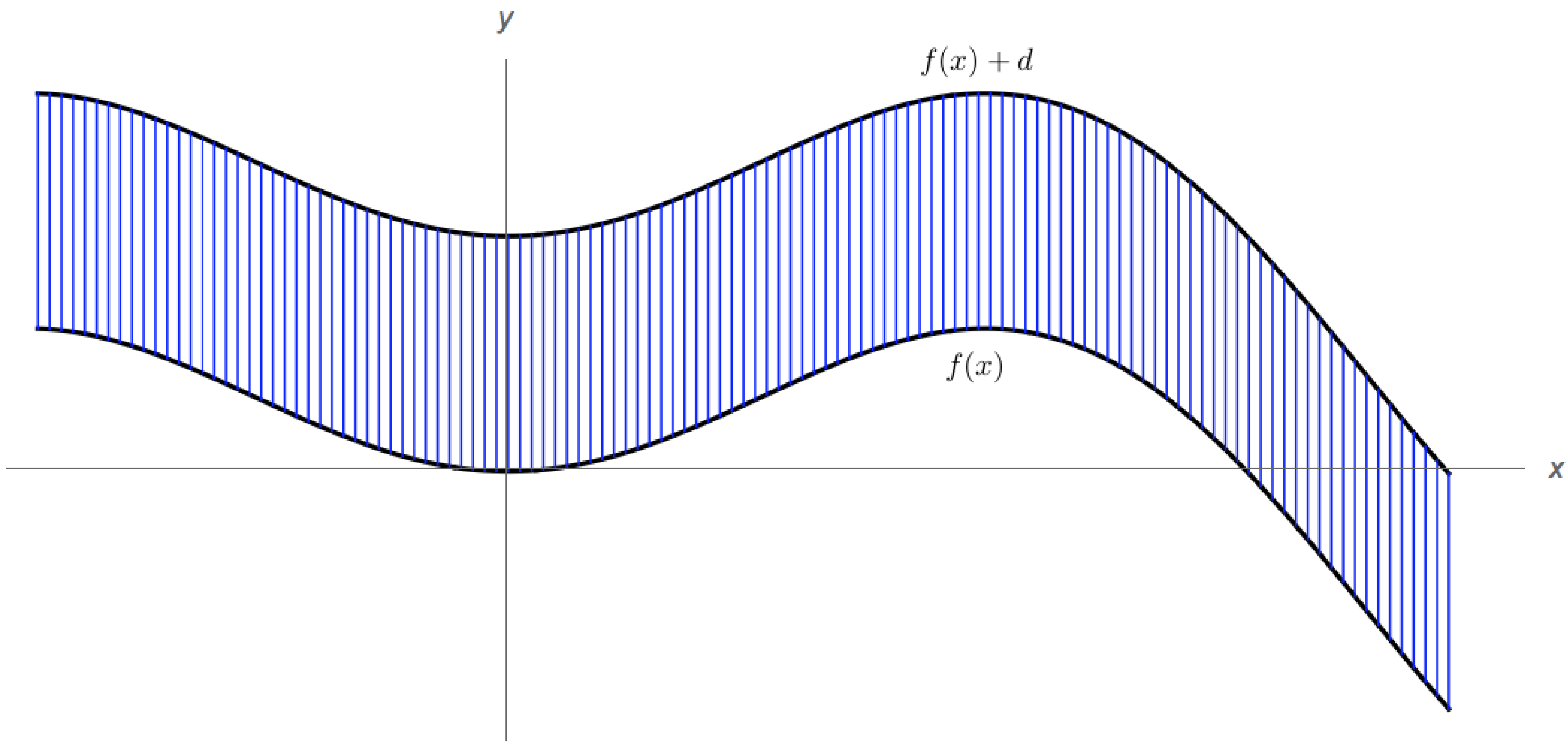


q.e.d.

[Barseghyan, Khrabustovskyi 2018]: a Berezin-type upper bound for the eigenvalue moments.

Part 2.

Sheared ribbons



Toy model for twisted waveguides

[Briet, D.K. 2018]

¡ $|\Omega| = \infty$!

$$\Omega := \left\{ \begin{pmatrix} s \\ f(s) + t \end{pmatrix} : (s,t) \in \mathbb{R} \times (0,d) \right\}$$

$f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ shear deformation

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Theorem (quasi-bounded realisation). If $\beta = \pm\infty$, then

$$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$$

(essential spectrum is empty)

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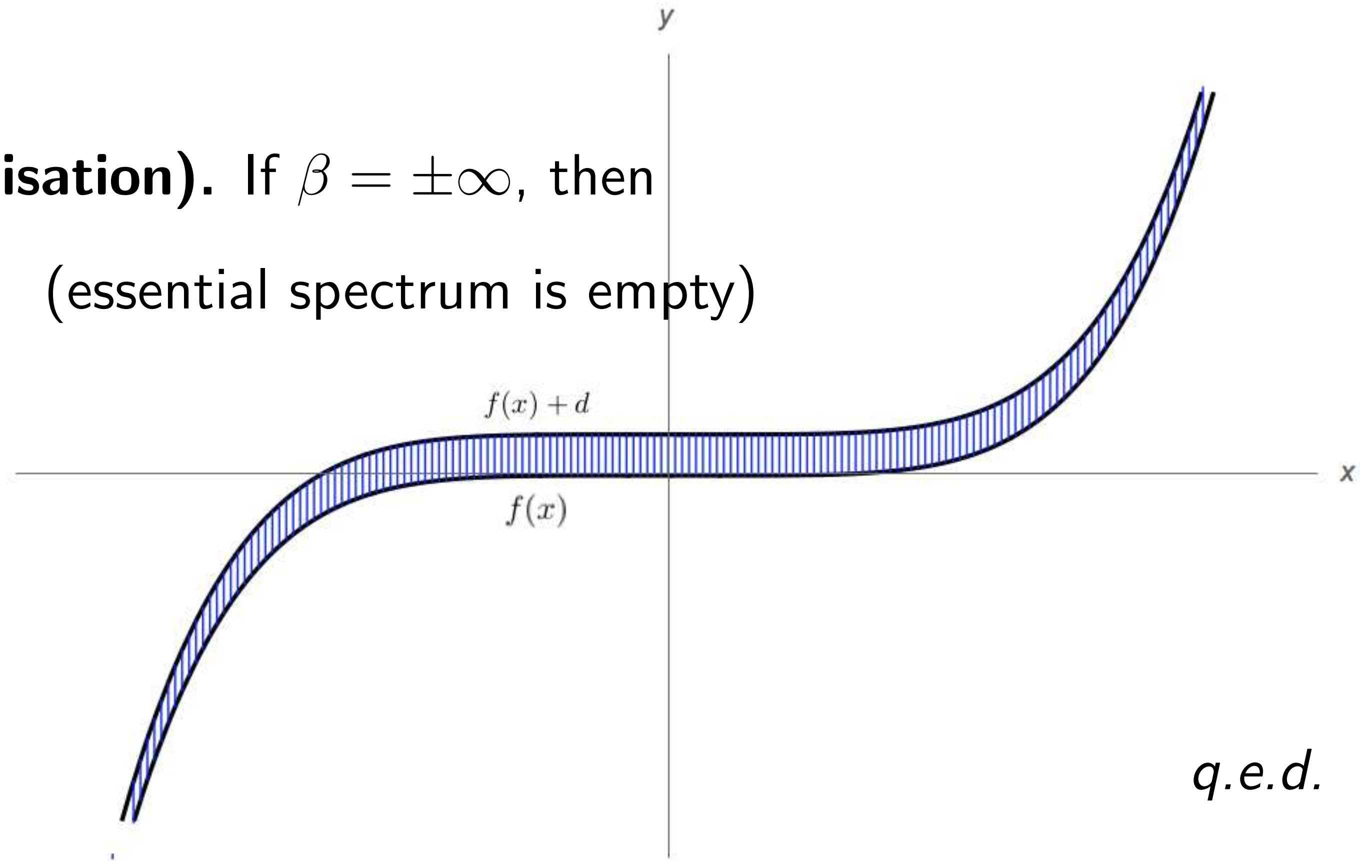
Proof.

$$\limsup_{\substack{|x|\to\infty \\ x\in\Omega}} |B_1(x) \cap \Omega| = 0$$

[Berger, Schechter 1972]



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Analogy with curved wedges

[D.K. *Portugal. Math.* 2016]

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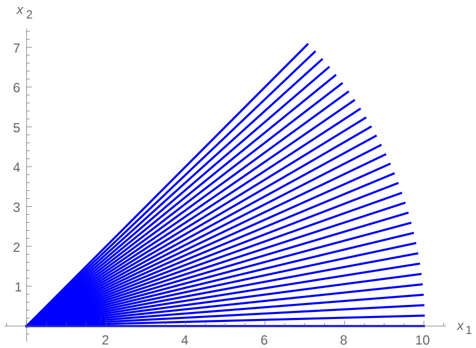
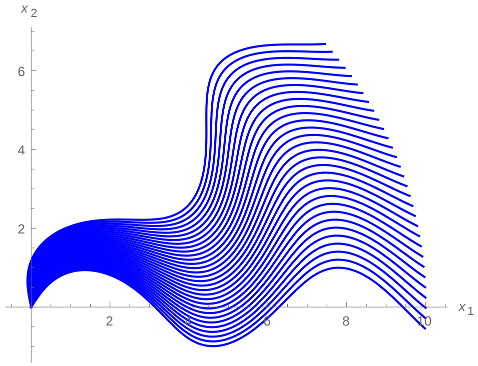
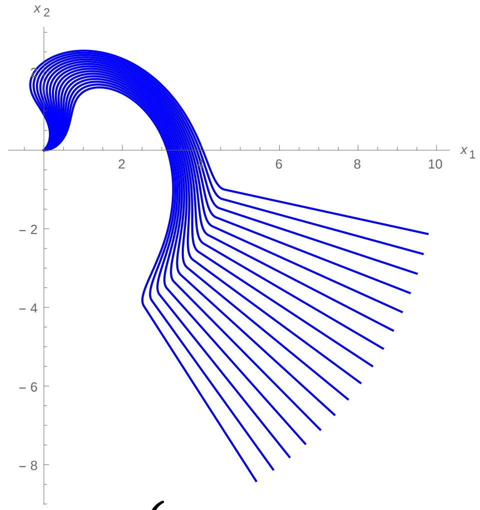


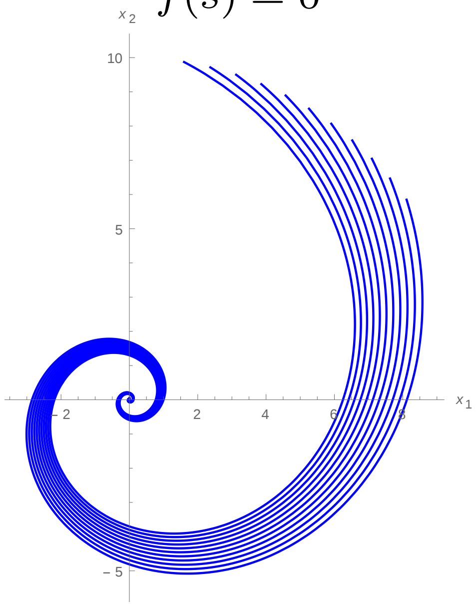
$f(s) = 0$



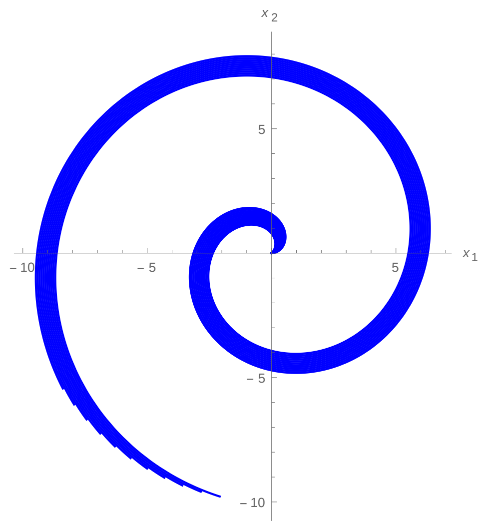
$f(s) = \sin(s)/s$



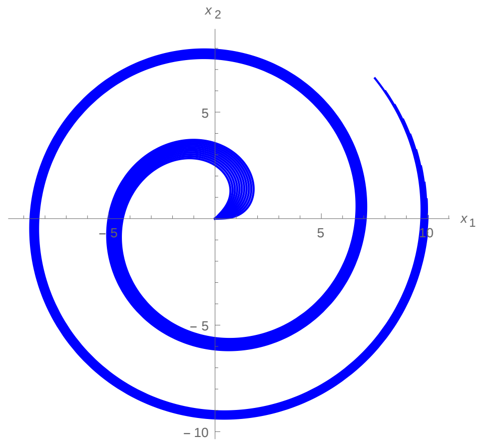
$$f(s) = \begin{cases} \sin(s), & s \le 3\pi/2 \\ -1, & s > 3\pi/2 \end{cases}$$



$f(s) = 3\log(s)$



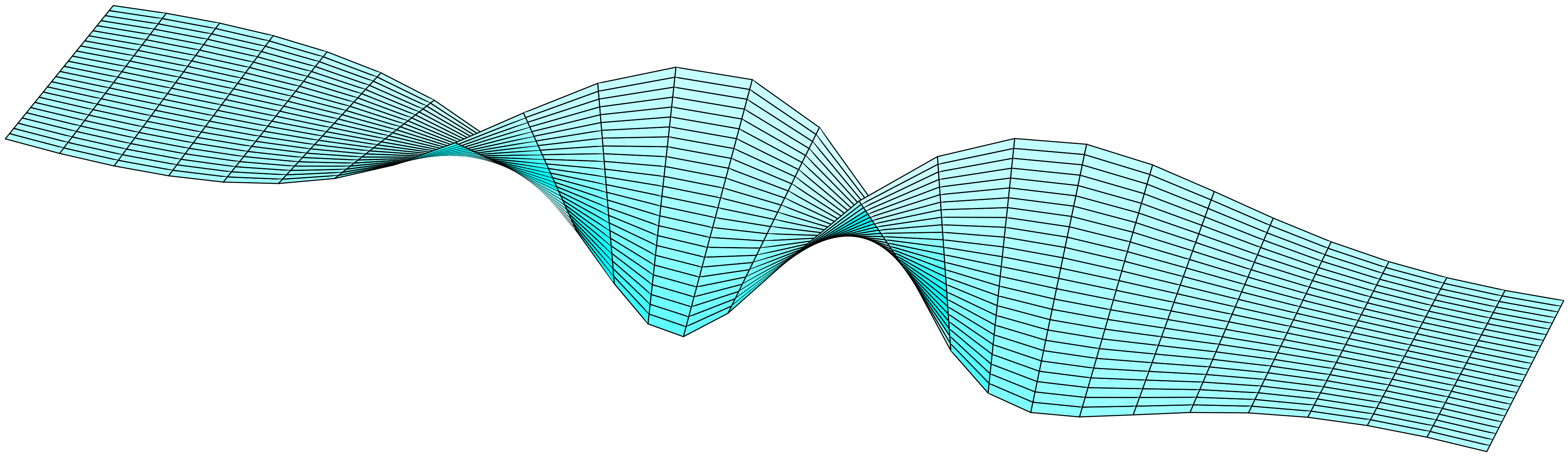
$f(s) = s$



$f(s) = s^2/8$

Part 3.

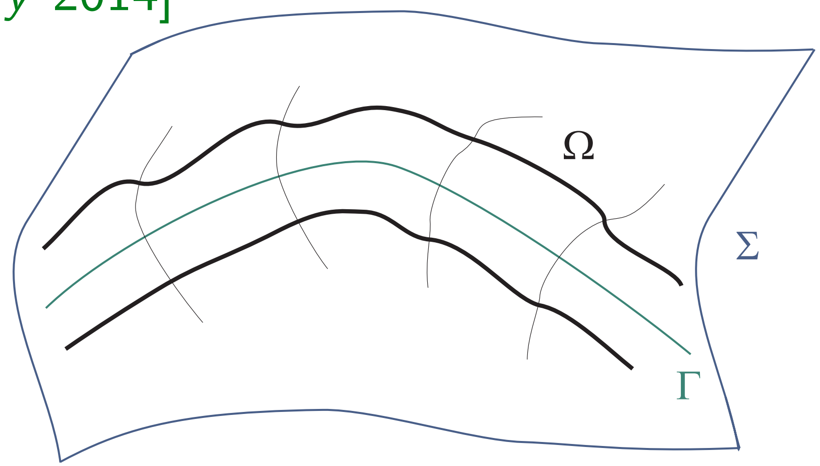
Ruled strips



The Brownian traveller on manifolds

[Kolb, D.K. *J. Spectr. Theory* 2014]

$\Omega :=$ tubular neighbourhood of an infinite geodesic Γ
on a *locally* deformed plane Σ of curvature K



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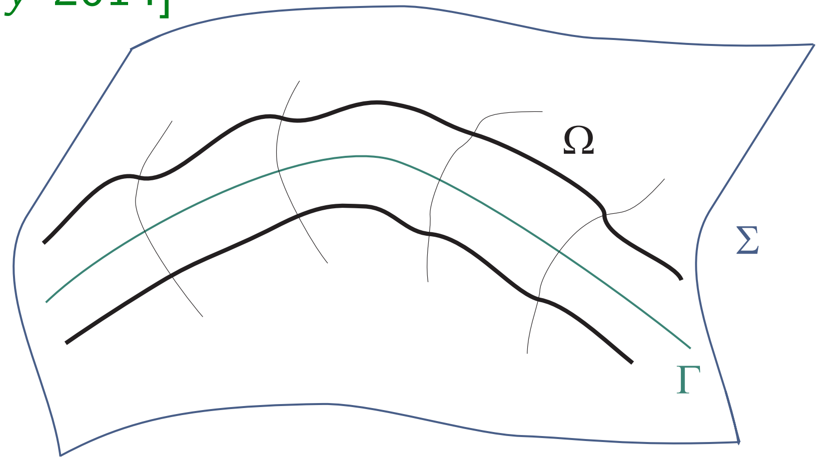
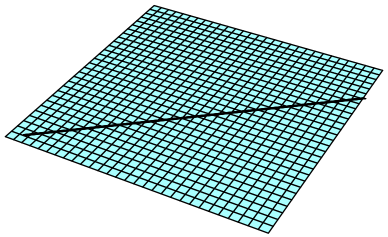
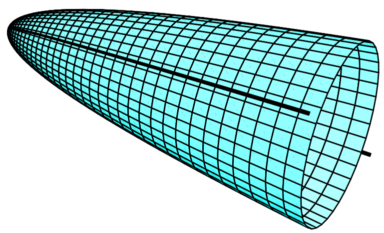
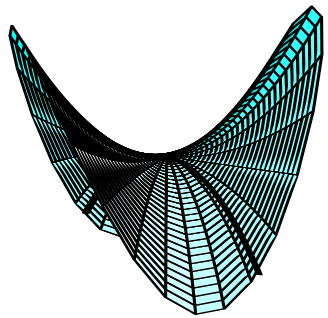


Ω	$K = 0$ (1800 ?)[1]	$K > 0$ (2003)[2]	$K < 0$ (2006)[3]
interaction	*null*	*attractive*	*repulsive*
spectrum			
math	criticality	bound states	Hardy inequalities







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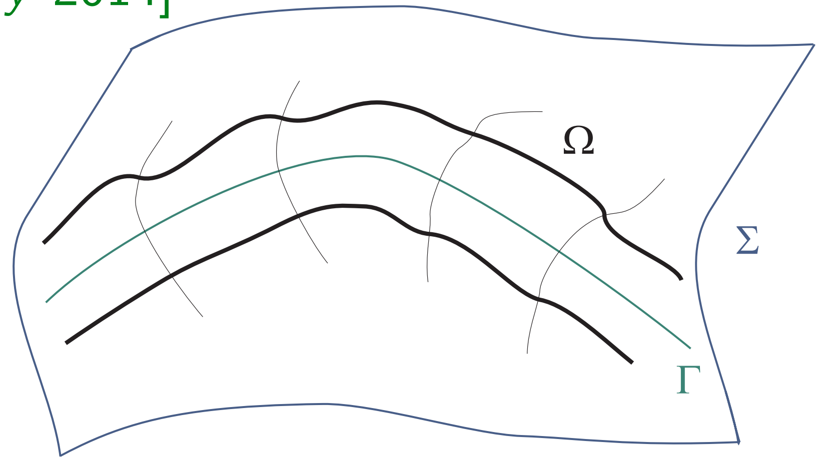

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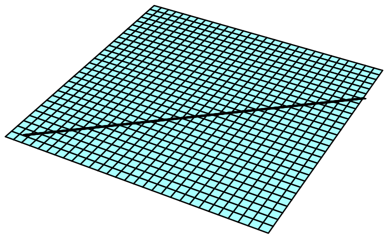
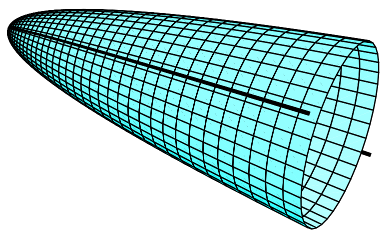
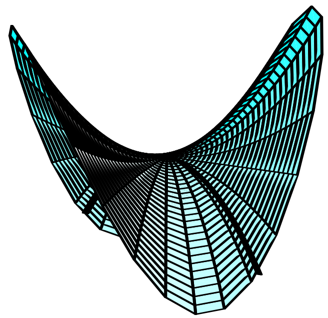
analogy	straight	bent	twisted
			

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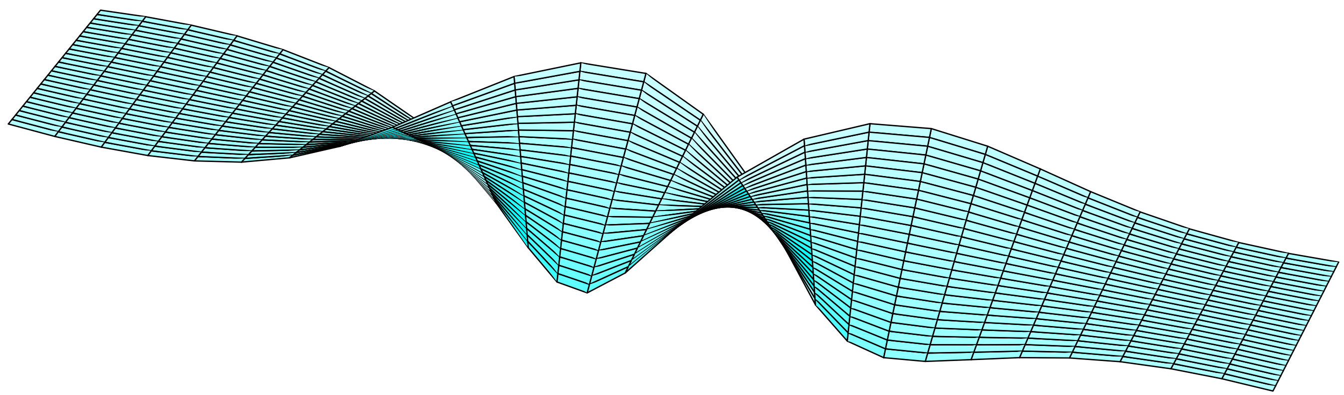
Negative curvature meets twisting

$$\Omega := \left\{ \begin{pmatrix} s \\ t\cos\theta(s) \\ t\sin\theta(s) \end{pmatrix} : (s,t) \in \mathbb{R} \times (a_1, a_2) \right\} \qquad \theta \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \qquad \text{twisting angle}$$



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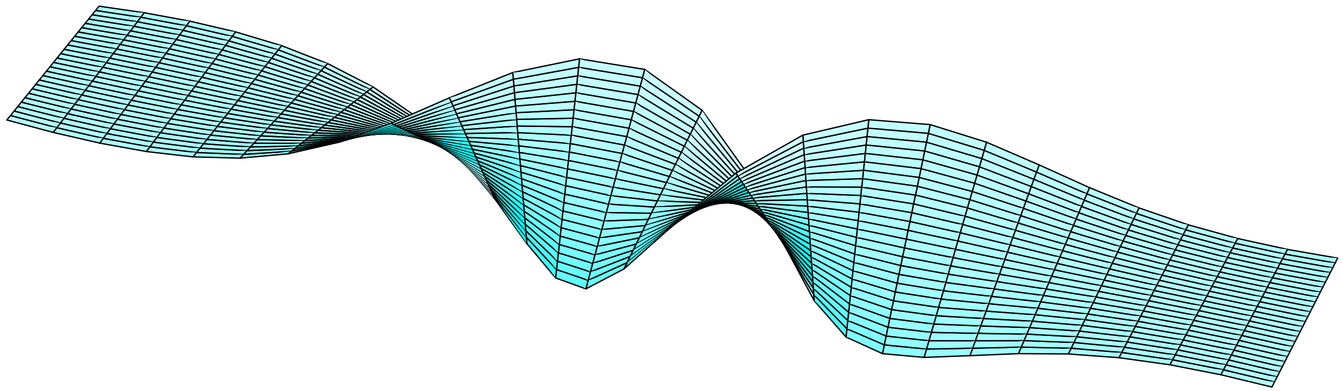

$$\Omega \cong \big(\mathbb{R} \times (a_1,a_2), G\big) \quad \text{with} \quad G(s,t) := \begin{pmatrix} 1+\theta'(s)^2\, t^2 & 0 \\ 0 & 1 \end{pmatrix}$$

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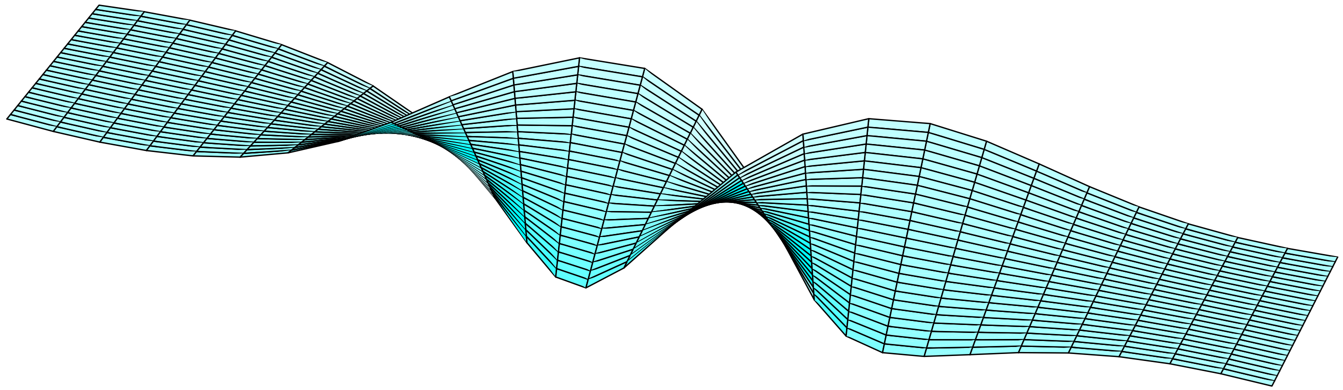

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Local versus global twisting

$$\theta'(s) \xrightarrow[|s|\to\infty]{} 0$$

$$\Longrightarrow \quad G \longrightarrow I$$

$$\Longrightarrow \quad -\Delta_D^\Omega \longrightarrow H_0 := -\partial_s^2 - \partial_t^2 \quad \text{in} \quad L^2(\mathbb{R} \times (a_1, a_2))$$

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raise of dimension

$$A_{a_1,a_2} := \left\{ \begin{pmatrix} t\cos\varphi \\ t\sin\varphi \end{pmatrix} : \ t\in(a_1,a_2),\ \varphi\in[0,2\pi) \right\}$$

Ruled strips with asymptotically diverging twisting

[D.K., Rafael Tiedra de Aldecoa *Ann. Henri Poincaré* 2018]

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Theorem 1. $\sigma_{\mathrm{ess}}(-\Delta_D^{\Omega}) = [\lambda_1, \infty)$

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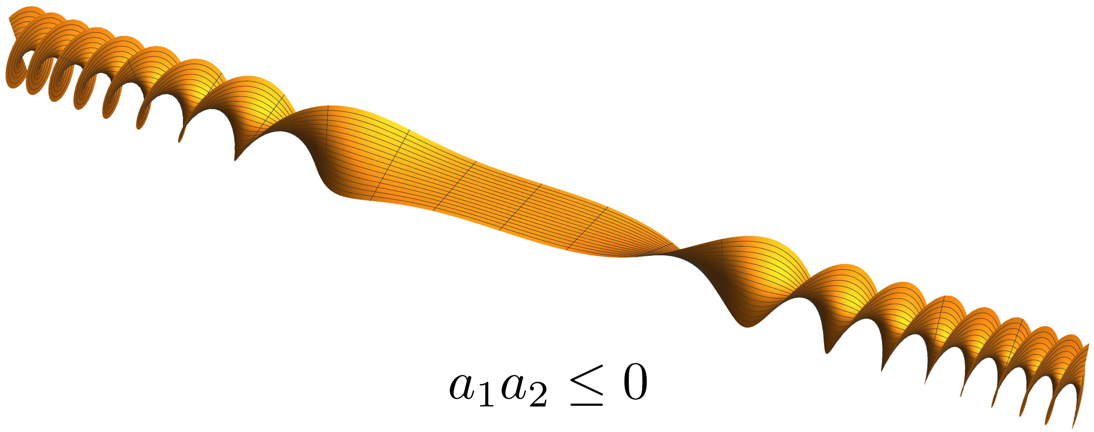
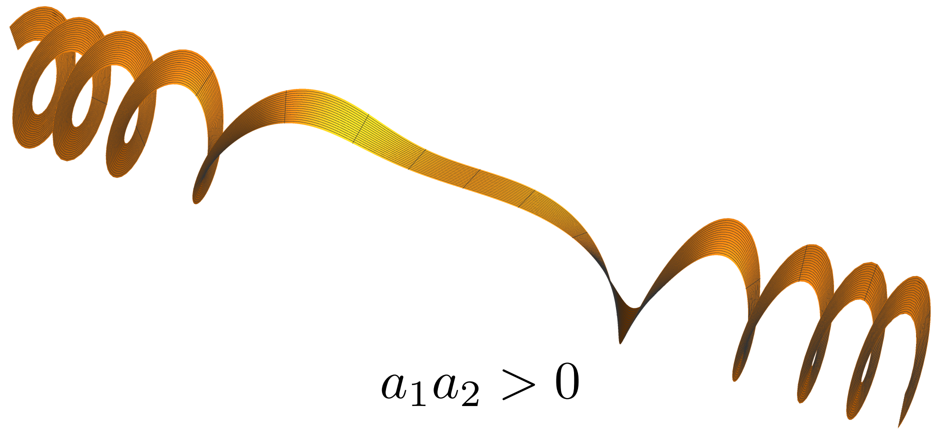


$a_1 a_2 \leq 0$



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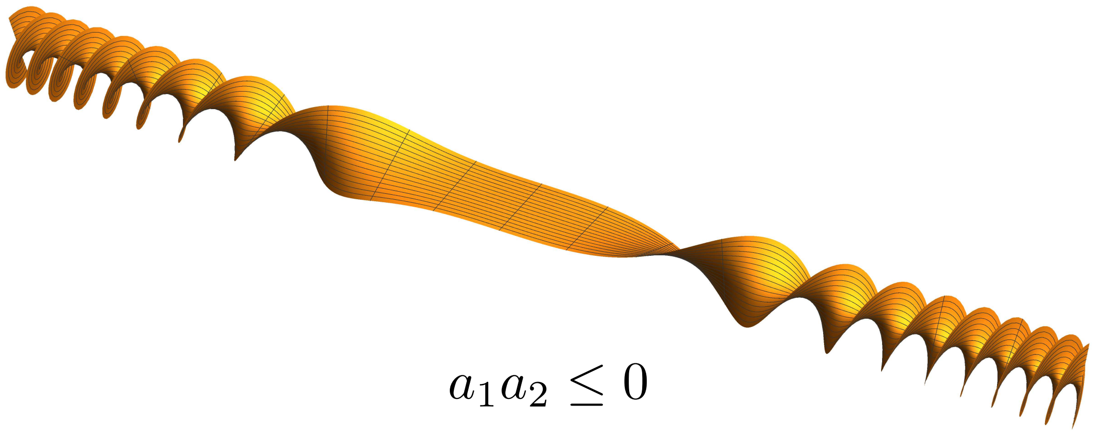

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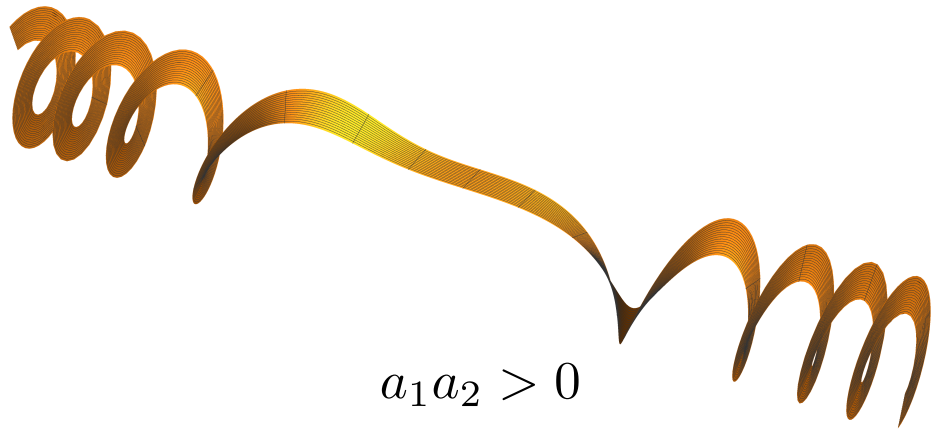

$a_1 a_2 > 0$

Theorem 2. If $a_1 a_2 \leq 0$ then $\min \sigma(-\Delta_D^\Omega) < \lambda_1.$ ($\exists$ bound states)

Part 4.

Conclusions

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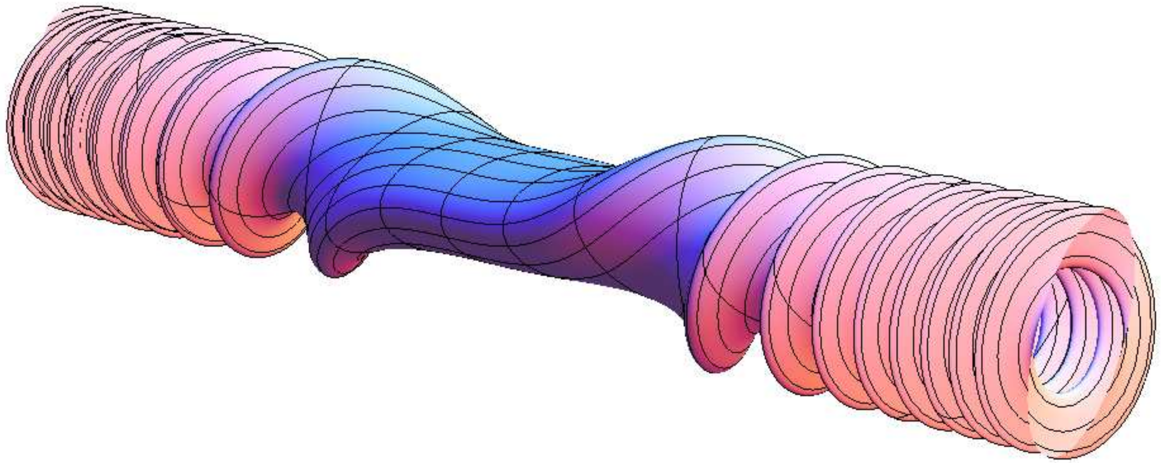
Summary

- New class of waveguide geometries: twisting diverging at infinity.

1. twisted tubes: both with and without essential spectra
2. sheared ribbons: always without essential spectra
3. ruled strips: always with essential spectra



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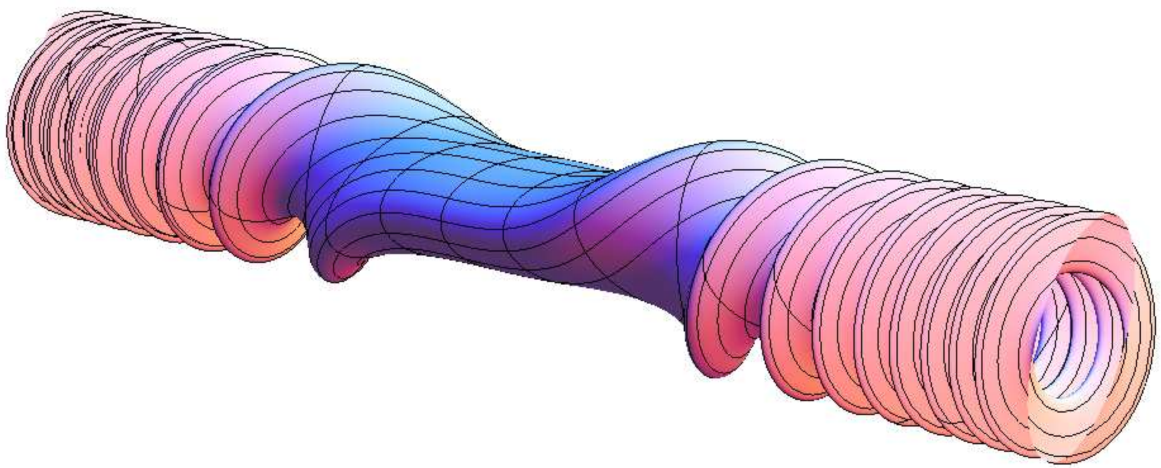

Open problems

¿ (non-standard) Weyl-type asymptotics **?** *cf.* [Barseghyan, Khrabustovskyi 2018]

¿ existence/number/properties of discrete eigenvalues (eigenfunctions) **?**

¿ nature of the essential spectrum **?** (embedded eigenvalues, Mourre theory, *etc.*)

¿ (reliable) numerics **?**