Quantum graphs and almost periodic functions

Pavel Kurasov
Jan Boman & Rune Suhr

(Stockholm)

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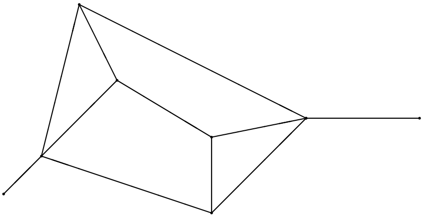
Graz

Quantum graph

- **Metric graph**



- **Differential expression** on the edges

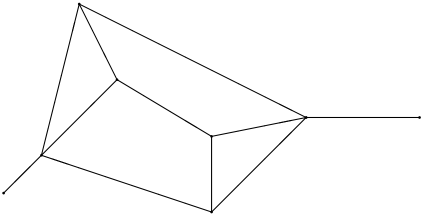
$$\ell_{q,a} = \left(i\frac{d}{dx} + a(x) \right)^2 + q(x)$$

- **Matching conditions**
  Via irreducible unitary matrices S^m associated with each internal vertex V_m

$$i(S^m - I)\vec{\psi}_m = (S^m + I)\partial\vec{\psi}_m, \;\; m = 1, 2, \ldots, M.$$

Quantum graph

- **Metric graph**



- **Differential expression** on the edges with zero magnetic potential

$$\ell = -\frac{d^2}{dx^2} + q(x)$$

- **Matching conditions**
  Via irreducible unitary matrices S^m associated with each internal vertex V_m

$$i(S^m - I)\vec{\psi}_m = (S^m + I)\partial\vec{\psi}_m, \ \ m = 1, 2, \dots, M.$$

Our assumption:

- The metric graph Γ is connected and formed by a finite number of compact edges.

Exceptional parameters

- **Single interval** $[0, \ell]$ as a metric graph
  The interval has the smallest Laplacian spectral gap among all graphs of the same total length.
- **Laplacian** $q(x) \equiv 0$
  $$-\frac{d^2}{dx^2}$$
  Zero potential is the only potential that can be determined *a priori* without knowing the metric graph.
- **Standard matching conditions**
  $$\begin{cases} \text{the function is continuous at } V_m, \\ \text{the sum of normal derivatives is zero.} \end{cases}$$
  Standard conditions appear if one requires continuity of the functions from the quadratic form domain. Easy to prescribe if nothing is known about the metric graph.

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Spectral properties

- Any Schrödinger operator with any vertex conditions is asymptotically isospectral to a Laplacian with scaling-invariant vertex conditions on essentially the same metric graph (+ Suhr)

$$k_n(L_q^S(\Gamma)) - k_n(L_0^{S^\infty}(\Gamma^\infty)) \to 0 \ \ \big(O(1/n)\big)$$

The approximating operator $L_0^{S^\infty}(\Gamma^\infty)$ is determined by:

  - the potential is zero $q \equiv 0$;
  - S^∞ are obtained from S by substituting all eigenvalues $\neq \pm 1$ with 1;
  - Γ^∞ is a graph obtained from Γ

Laplacian with scaling-invariant vertex conditions:

  - $q \equiv 0 \Rightarrow$ eigenfunctions are given by exponentials on the edges;
  - the vertex conditions are determined by S^∞ - unitary and Hermitian $\Rightarrow$ the vertex scattering matrices are energy-independent

$\Rightarrow$ the spectrum is given by zeroes of trigonometric polynomials

$$P(k) = \sum_{j \in J} a_j e^{i w_j k}.$$

Conclusion The theory of almost periodic functions can be applied to describe spectral asymptotics.

- The spectrum $\lambda_n = k_n^2$ is discrete and satisfies Weyl's asymptotics

$$\boxed{k_n \sim \frac{\pi}{\mathcal{L}} n} \quad \mathcal{L} - \text{the total length of the graph.}$$

NB! No further asymptotic expansion is available:

$$k_n = \frac{\pi}{\mathcal{L}} n + c_0 + c_{-1} \frac{1}{n} + \dots$$

Two exceptions:

- Equilateral graphs: the spectrum of the Laplacian with scaling-invariant vertex conditions is periodic (in k). The Schrödinger asymptotics: $\exists N \in \mathbb{N}$

$$k_n = \underbrace{\frac{\pi}{\mathcal{L}} \left[\frac{n}{N} \right] N + k_{\{\frac{n}{N}\}N}}_{= \frac{\pi}{\mathcal{L}} n + O(1)} + O(1/n), \quad j = 1, 2, \dots, N.$$

  The Laplacian spectrum is uniformly discrete, but multiple eigenvalues may occur.

  In general situation the spectrum of the scaling-invariant (or standard) Laplacian is not necessarily uniformly discrete.

- Weak vertex couplings: all matrices S^m in the vertex conditions do not have -1 as an eigenvalue. The spectrum is approximated by the spectra of Neumann Laplacians on disconnected intervals. (Freitas-Lipovsky)

To solve the inverse problem one has to reconstruct all three members of the quantum graph triple

- the metric graph Γ;
- the real potential $q(x) \in L_1(\Gamma)$;
- the vertex conditions, *i.e.* the matrices S^m.

This problem is not solvable if the spectral data are just the eigenvalues of the quantum graph:

- isospectral standard Laplacians on trees;
- potential on a single interval is determined by two spectra: Neuman-Neuman and Neumann-Dirichlet;
- standard Laplacian on a single interval is isospectral to the union of half-intrevals with Nuemann-Neumann and Dirichlet-Neumann conditions.

One exception: Ambartsumian theorem.

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One exception: Ambartsumian theorem.

Our goal is to investigate the inverse spectral problem when the quantum graph does not differ much from the Laplacian on a single interval (the parameters do not differ from the exceptional ones).

We start by investigating the inverse problem when two parameters are fixed and one is varying

- Potential varyes (graph-interval, standard conditions)
- Graph varies (potential zero, standard conditions)
- Vertex conditions vary (graph-interval, potential zero)

and continue to the case, where just one parameter is fixed

- Standard vertex conditions fixed (graph and potential vary)
- Graph-interval is fixed (potential and conditions vary)
- Potential is fixed - Laplace operator (graph and vertex conditions vary)

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Viktor Ambratsumian (1908-1996)

Worked at Pulkovo (Pulkowo) observatory
Vice-rector of Leningrad Univ.
President of Armenian Academy of Sciences 1947-1993.

Our goal is to prove an Ambartsumian type theorem for quantum graphs

Interval with standard conditions

Classical Ambartsumian theorem

$I = [0, \ell]$, $q \in L_1(I)$

$$\lambda_n(L_q^{\mathrm{st}}(I)) = \lambda_n(L_0^{\mathrm{st}}(I) \Rightarrow q(x) \equiv 0$$

Proof

1. Spectral asymptotics using transformation operator

$$k_n(L_q^{\mathrm{st}}(I)) = \frac{\pi}{\ell} \left(n + \frac{\int_0^\ell q(x)dx}{\ell} \frac{1}{2} \left(\frac{\ell}{\pi} \right)^2 \frac{1}{n} + o(1/n) \right)$$

2. The trial function $\psi(x) \equiv 1$ minimises the quadratic form

$$\int_0^\ell |\psi'(x)| dx + \int_0^\ell q(x) |\psi(x)|^2 dx = 0$$

and therefore is the ground state

$$-\psi''(x) + q(x)\psi(x) = 0 \Rightarrow q(x) \equiv 0.$$

Zero potential is exceptional

Standard Laplacian on arbitrary metric graph

Geometric version of Ambartsumian theorem

Two may be different metric graphs $I = [0, \ell]$ and Γ; $q(x) \equiv 0$

$$\lambda_n(L_0^{\mathrm{st}}(\Gamma)) = \lambda_n(L_0^{\mathrm{st}}(I) \Rightarrow \Gamma = I$$

(Nicaise, Friedlander, B.Solomjak, Kurasov-Naboko, KKMB, ...)

Proof The interval minimises the Laplacian spectral gap among all metric graphs of the same total length.

Eulerian path technique: Double the graph. All vertices have even degree, There exists an Eulerian path, Cut the doubled graph into a circle. The circle of double length has the same spectral gap as the interval. The spectral gap could just become smaller during our changes.

The interval is exceptional

Laplacian on the interval

Laplacian with Robin conditions on the interval
Let $L_0^{\mathbf{h}}(I)$ be a Robin Laplacian on the interval $I = [0, \ell]$.

$$\lambda_n(L_0^{\mathbf{h}'}(I)) = \lambda_n(L_0^{\mathbf{h}}(I)) \Rightarrow \mathbf{h}' = \mathbf{h}$$

(Of course, up to the permutation of the end points)

The standard conditions are not necessarily exceptional

We continue by investigating the inverse problem when just one parameter is fixed and two are varying

- Standard vertex conditions fixed (graph and potential vary)
- Graph-interval is fixed (potential and conditions vary)
- Potential is fixed - Laplace operator (graph and vertex conditions vary)

Schrödinger operators on arbitrary graphs with standard vertex conditions

Theorem (Boman-K.-Suhr)

$$\lambda_n(L_q^{\rm st}(\Gamma)) = \lambda_n(L_0^{\rm st}(I)) \Rightarrow \left\{ \begin{array}{l} \Gamma = I \\ q(x) \equiv 0. \end{array} \right.$$

This theorem is **not** a simple combination of the classical Ambartsumian theorem and its geometric version.

Proof

1. The spectrum of $L_q^{\rm st}(\Gamma)$ is asymptotically close to the spectrum of $L_0^{\rm st}(\Gamma)$.
2. The spectrum of the Laplacian is given by a trigonometric polynomial and therefore is close to integers if and only if it coincide with the integers.
3. Geometric version of Ambratsumian theorem implies that the graph Γ is just the interval.
4. Classical Ambarstumian theorem implies that $q(x) \equiv 0$.

The standard conditions are exceptional

Schrödinger operators with Robin conditions on the interval

Use Crum's article where Darboux transform was used to add eigenvalues to the Schrödinger operator on the interval.
Start with the Dirichlet Laplacian: $\lambda_n = \left(\frac{\pi n}{\ell}\right)^2$. Its spectrum differs from the Neumann Laplacian by just one eigenvalue $\lambda = 0$. We add this eigenvalue by Crum's method
Interval $[0, 1]$

$$q(x) = \frac{-1}{x+1}, h_0 = -1, h_1 = \frac{1}{2}$$

$$\psi_1(x) = \frac{1}{x+1}$$

$$\psi_{n+1} = -\frac{1}{\pi^2 n^2}\left(n \cos nx - \frac{\sin nx}{x+1}\right)$$

We constructed a family of Robin Schrödinger operators isospectral to the Neumann Laplacian $\Rightarrow$ no Ambartsumian-type theorem.

Laplace operators on arbitrary graphs with arbitrary vertex conditions

Γ - the two intervals of length $1/2$ connected at one vertex.
We assume standard (=Neumann) conditions at the outer vertices. Conditions at the central vertex are given as

$$i(S-I)\begin{pmatrix} u(x_1) \\ u(x_2) \end{pmatrix} = (S+I)\begin{pmatrix} \partial u(x_1) \\ \partial u(x_2) \end{pmatrix}$$

with the 2×2 matrix S unitary and Hermitian

$$S^{-1} = S^* = S.$$

Then it holds:

$$\lambda_n(L_0^{S,\mathrm{st}}(\Gamma)) = \lambda_n(L_0^{\mathrm{st}}(I))$$

The proof is essentially based on the fact that $S^2 = \mathbb{I}$. and explicit calculation of the ground state.

Every such 2×2 matrix possess the representation:

$$S(a,\theta) = \begin{pmatrix} a & \sqrt{1-a^2}e^{i\theta} \\ \sqrt{1-a^2}e^{-i\theta} & -a \end{pmatrix}$$

This family interpolates between the case of single interval (standard vertex conditions at the central vertex) and two intervals with Dirichlet and Neumann conditions

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

no Ambartsumian-type theorem

Further extensions of Ambartsumian theorem

- A theorem by Brian Davies
- A theorem by P.K. and Rune Suhr and its implications

A theorem by Brian Davies

Theorem

The metric graph is arbitrary but fixed.

$$\lambda_n(L_q^{\rm st}(\Gamma)) = \lambda_n(L_0^{\rm st}(\Gamma)) \Rightarrow q(x) \equiv 0$$

Proof

1. $\lambda_1(L_q^{\rm st}(\Gamma)) = 0$ & $\int q(x)dx = 0 \Rightarrow q(x) \equiv 0$

- the same proof as before would show that the potential is zero.

2. Let H_Γ be the heat kernel for $L_0^{\rm st}(\Gamma)$

$$\lim_{t\to 0} \sqrt{t} H_\Gamma(t,x,x) = \frac{1}{\sqrt{4\pi}}, \;\; x \in \Gamma \setminus (\cup_{m=1}^M V_m)$$

3. Perturbation formula for traces of the semigroups

$$\mathrm{tr}\,[e^{-L_q^{\rm st}(\Gamma)t}] - \mathrm{tr}\,[e^{-L_0^{\rm st}(\Gamma)t}] = -t \int_\Gamma H_\Gamma(t,x,x) q(x) dx + \rho(t),$$

where $\rho(t) = O(t^{3/2})$.

A theorem by P.K. and Rune Suhr

Theorem Assume that the Laplacians on Γ_1 and Γ_2 are asymptotically isospectral

$$k_n(L_0^{\mathrm{st}}(\Gamma_1)) - k_n(L_0^{\mathrm{st}}(\Gamma_2)) \to 0$$

then the operators are isospectral.

Proof

1. The spectrum of Laplacians with standard vertex conditions is given by zeroes of trigonometric polynomials, which are analytic almost periodic functions.

2. If the zeroes of two almost periodic functions are asymptotically close, then they coincide. The proof is based on the existence of ϵ-shifts.

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Implications of the two theorems

Theorem 1
Two Schrödinger operators $L^{\rm st}_{q_1}(\Gamma_1)$ and $L^{\rm st}_{q_2}(\Gamma_2)$ are asymptotically isospectral if and only if the Laplacians

$$L^{\rm st}_0(\Gamma_1) \text{ and } L^{\rm st}_0(\Gamma_2)$$

are isospectral.
Proof: spectral asymptotics + Theorem by PK and RS

$$\lambda_n(L^{\rm st}_q(\Gamma) - \lambda_n(L^{\rm st}_0(\Gamma) = O(1)$$

$$\Rightarrow k_n(L^{\rm st}_q(\Gamma) - k_n(L^{\rm st}_0(\Gamma) = o(1)$$

Implications of the two theorems

Theorem 2
A Schrödinger operator $L_{q_1}^{\mathrm{st}}(\Gamma_1)$ is isospectral to a Laplacian $L_0^{\mathrm{st}}(\Gamma_2)$ only if

$$q(x) \equiv 0.$$

Proof
1. The Laplacian $L_0^{\mathrm{st}}(\Gamma_1)$ is asymptotically isospectral to $L_0^{\mathrm{st}}(\Gamma_2)$ $\Rightarrow$ they are isospectral.

2. The Schrödinger and Laplace operators on Γ_1 are isospectral $\Rightarrow$ the theorem by Davies implies $q \equiv 0$.

Higher order operators on metric graphs

- First order operator: momentum operator $\frac{1}{i}\frac{d}{dx}$ or the Dirac operator $\mathrm{diag}(\frac{1}{i}\frac{d}{dx}, -\frac{1}{i}\frac{d}{dx})$.
  The vertex conditions are given by unitary matrices ($d/2 \times d/2$ or $d \times d$), the vertex scattering matrices are always independent of the energy.
  The spectrum of every such operator is given by a trigonometric polynomial;
- Second order operator: Laplacian $-\frac{d^2}{dx^2}$.
  The vertex conditions are given by $d \times d$ unitary matrices, the vertex scattering matrices tend to Hermitian unitary matrices for large energies.
  The spectrum is asymptotically close to zeroes of a trigonometric polynomial, corresponding to a certain scaling-invariant Laplacian on essentially the same metric graph.
- Fourth order operator: bi-Laplacian $\frac{d^4}{dx^4}$.
  The vertex conditions may be given by $2d \times 2d$ transmission matrices, having vertex scattering matrices as a $d \times d$ block.
  The spectrum is asymptotically close to zeroes of a trigonometric polynomial, corresponding to a certain Dirac operator on the same metric graph.

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