Algebra of waves

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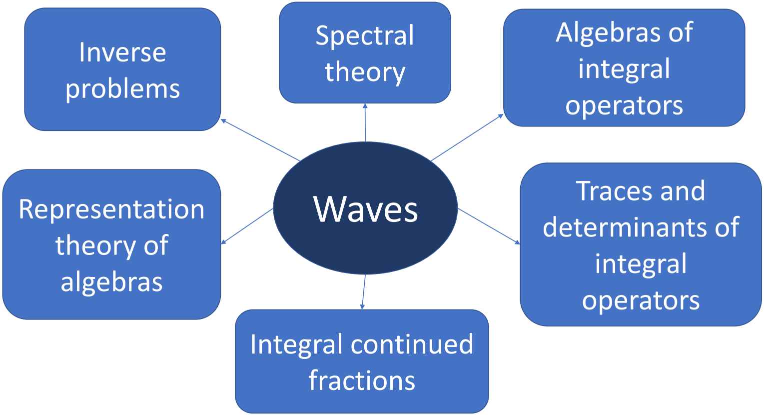
February 25, 2019

Some aspects of mathematical theory of waves



What are waves, their periods, amplitudes, etc.?

Waves are functions

In general, waves are oscillations

$$\mathbf{u}(\mathbf{x},t) = e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\mathbf{u}_0(\mathbf{x}),$$

where ω is a frequency, $\mathbf{k}$ is a wave-number, and $\mathbf{u}_0(\mathbf{x})$ is 1-periodic function along the direction of the wave propagation (along the vector $\mathbf{k}$). Wave length (space-period) L, time-period T, and amplitude U are

$$L = 2\pi/|\mathbf{k}|, \quad T = 2\pi/\omega, \quad U = \max|\mathbf{u}_0|.$$

1) If $\mathbf{u}_0(\mathbf{x})$ is periodic along the wave propagation and bounded, non-decreasing along other directions then $\mathbf{u}(\mathbf{x},t)$ is a volume wave.
2) If $\mathbf{u}_0(\mathbf{x})$ is periodic along the wave propagation and decreasing along other directions then $\mathbf{u}(\mathbf{x},t)$ is a guided wave.

Dispersion diagrams (spectrum)

Usually, for a given wave

$$\mathbf{u}(\mathbf{x}, t) = e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\mathbf{u}_0(\mathbf{x}),$$

the parameters ω, $\mathbf{k}$, and $\mathbf{u}_0(\mathbf{x})$ are related by certain equations

$$\omega = \omega(\mathbf{k}), \quad \mathbf{u}_0 = \mathbf{u}_0[\mathbf{k}, \omega].$$

To find them, we should substitute $\mathbf{u}$ into the wave equation

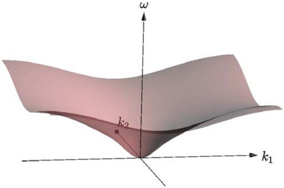
$$\ddot{\mathbf{u}} = \mathcal{A}\mathbf{u},$$

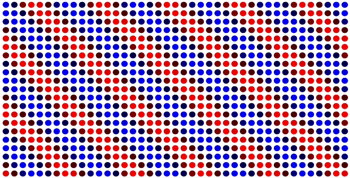
where $\mathcal{A}$ is some periodic operator, e.g. $\mathcal{A} = \rho^{-1}\nabla \cdot \mu\nabla$ or discrete $\mathcal{A}u_n = \rho^{-1}\sum_{n\sim n'} \mu_{n'}(u_{n'} - u_n)$, etc.. After substitution

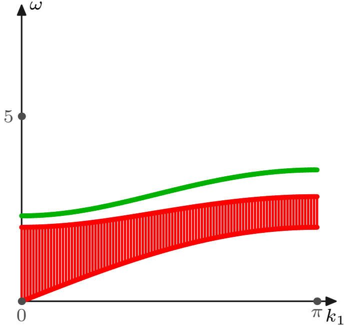
$$-\omega^2\mathbf{u}_0 = \mathcal{A}_{\mathbf{k}}\mathbf{u}_0.$$

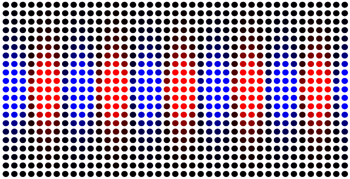
Hence, $\omega^2 = \omega^2(\mathbf{k})$ are "eigenvalues", and $\mathbf{u}_0 = \mathbf{u}_0[\mathbf{k}, \omega]$ are corresponding "eigenvectors" of $-\mathcal{A}_{\mathbf{k}}$.

Volume, guided waves and dispersion diagrams





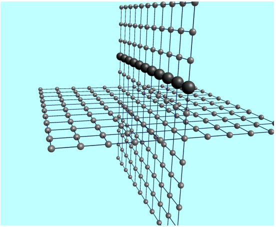


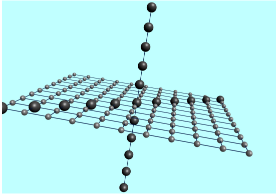


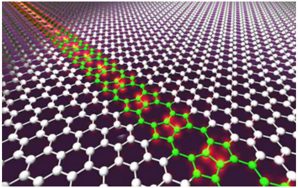
Usually, we can not observe guided (local) waves in uniform and purely periodic structures.

To observe them we should consider periodic structures with embedded defects of lower dimension.

Example of periodic lattices with defects







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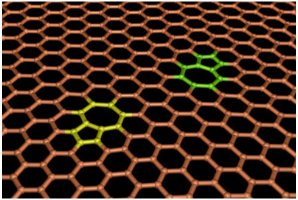

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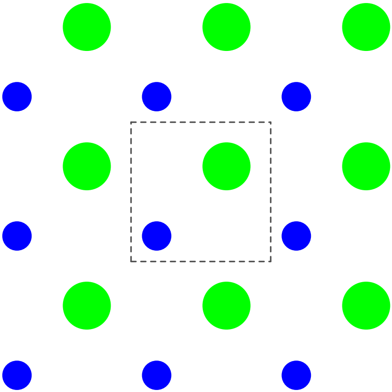
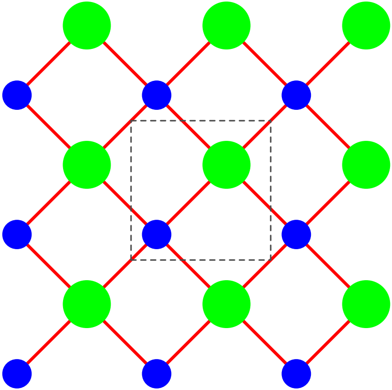
Periodic lattices



We can define N-periodic lattice with M-point unit cell as follows

$$\Gamma = [1, ..., M] \times \mathbb{Z}^N.$$



Any (bounded) operator

$$\mathcal{A} : \ell^2(\Gamma) \to \ell^2(\Gamma)$$

which commutes with all shift operators

$$\mathcal{S}_{\mathbf{m}} u(j, \mathbf{n}) = u(j, \mathbf{n}+\mathbf{m}), \quad u \in \ell^2(\Gamma)$$

is called a periodic operator.

The corresponding transformation based on Fourier series

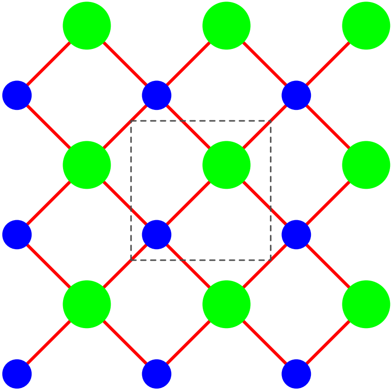
$$\mathcal{F} : \ell^2(\Gamma) \to L^2_{N,M} := L^2([0,1]^N \to \mathbb{C}^M),$$

$$(\mathcal{F}u)_j(\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^N} e^{2\pi i \mathbf{k}\cdot\mathbf{n}} u(j,\mathbf{n})$$

allows us to rewrite our periodic operator $\mathcal{A}$ as an operator of multiplication by a matrix-valued function $\mathbf{A}$

$$\hat{\mathcal{A}} := \mathcal{F}\mathcal{A}\mathcal{F}^{-1} : L^2_{N,M} \to L^2_{N,M}, \qquad \hat{\mathcal{A}}\mathbf{u} = \mathbf{A}\mathbf{u}.$$

Periodic operators after F-F-B transform



A periodic operator $\mathcal{A}$ unitarily equivalent to the following operator

$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

$$\hat{\mathcal{A}}\mathbf{u}(\mathbf{k}) = \mathbf{A}_0(\mathbf{k})\mathbf{u}(\mathbf{k})$$

with some (usually continuous) $M \times M$ matrix-valued function $\mathbf{A}_0(\mathbf{k})$ depending on the "quasi-momentum" $\mathbf{k} \in [0,1]^N$.

Spectrum of periodic operators

For the operator of multiplication by the matrix-valued function

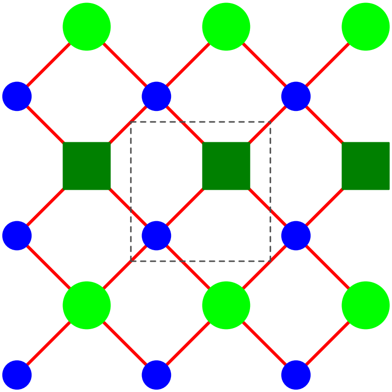
$$\hat{\mathcal{A}}\mathbf{u}(\mathbf{k}) = \mathbf{A}_0(\mathbf{k})\mathbf{u}(\mathbf{k})$$

the spectrum is just eigenvalues of this matrix for different quasi-momentums

$$\mathrm{sp}(\hat{\mathcal{A}}) = \{\lambda : \ \det(\mathbf{A}_0(\mathbf{k}) - \lambda \mathbf{I}) = 0 \ \textit{for some } \mathbf{k}\} =$$

$$\bigcup_{j=1}^{M} \bigcup_{\mathbf{k}\in[0,1]^N} \{\lambda_j(\mathbf{k})\}.$$

Periodic operators with linear defects ($N = 2$)



In this case our periodic operator

$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

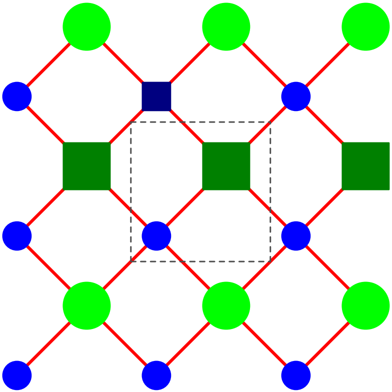
takes the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle \mathbf{B}_1\mathbf{u}\rangle_1$$

with some (usually continuous) matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ and

$$\langle\cdot\rangle_1 := \int_0^1 \cdot\, dk_1.$$

Periodic operators with linear and point defects ($N = 2$)



In this case our periodic operator

$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

takes the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle\mathbf{B}_1\mathbf{u}\rangle_1 + \mathbf{A}_2\langle\mathbf{B}_2\mathbf{u}\rangle_2$$

with some (usually continuous) matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ and

$$\langle\cdot\rangle_2 := \int_0^1\int_0^1 \cdot\, dk_1 dk_2.$$

Periodic operator with defects (general case)

In general, a periodic operator with defects is unitarily equivalent to the operator $\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M}$ of the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle \mathbf{B}_1\mathbf{u}\rangle_1 + ... + \mathbf{A}_N\langle \mathbf{B}_N\mathbf{u}\rangle_N.$$

with continuous matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ and

$$\langle\cdot\rangle_1 = \int_0^1 \cdot dk_1, \quad \langle\cdot\rangle_{j+1} = \int_0^1 \langle\cdot\rangle_j dk_{j+1}.$$

Remark. For simplicity we will write $\mathcal{A}$ instead of $\hat{\mathcal{A}}$. The spectrum of this operator is

$$\mathrm{sp}(\mathcal{A}) = \{\lambda : \ \mathcal{A} - \lambda\mathcal{I} \ is\ non-invertible\} = \{\lambda : \ \widetilde{\mathcal{A}} \ is\ non-invertible\},$$

where $\widetilde{\mathcal{A}}$ has the same form as $\mathcal{A}$ but with $\mathbf{A}_0 - \lambda\mathbf{I}$ instead of $\mathbf{A}_0$.

Test for invertibility of a periodic operator with defects

$$\mathcal{A} = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1 + \ldots + \mathbf{A}_N \langle \mathbf{B}_N \cdot \rangle_N$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_0^{-1} \mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1 + \ldots + \mathbf{A}_0^{-1} \mathbf{A}_N \langle \mathbf{B}_N \cdot \rangle_N)$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1 + \ldots + \mathbf{A}_{N0} \langle \mathbf{B}_N \cdot \rangle_N)$$

$$= (\mathbf{A}_0 \cdot) \underbrace{(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{I} - \mathbf{A}_{10} (\mathbf{I} + \langle \mathbf{B}_1 \mathbf{A}_{10} \rangle_1)^{-1} \langle \mathbf{B}_1 \cdot \rangle_1)}_{=\mathbf{I}} (\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1 + \ldots + \mathbf{A}_{N0} \langle \mathbf{B}_N \cdot \rangle_N)$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{I} + \mathbf{A}_{21} \langle \mathbf{B}_2 \cdot \rangle_2 + \ldots + \mathbf{A}_{N1} \langle \mathbf{B}_N \cdot \rangle_N)$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{I} + \mathbf{A}_{21} \langle \mathbf{B}_2 \cdot \rangle_2)(\mathbf{I} + \mathbf{A}_{32} \langle \mathbf{B}_3 \cdot \rangle_3 + \ldots + \mathbf{A}_{N2} \langle \mathbf{B}_N \cdot \rangle_N)$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{I} + \mathbf{A}_{21} \langle \mathbf{B}_2 \cdot \rangle_2) \ldots (\mathbf{I} + \mathbf{A}_{N,N-1} \langle \mathbf{B}_N \cdot \rangle_N)$$

Test for invertibility of a periodic operator with defects

Theorem (from J. Math. Anal. Appl., 2015)

Step 0. Define $\pi_0 = \det \mathbf{E}_0, \quad \mathbf{E}_0 = \mathbf{A}_0.$

If $\pi_0(\mathbf{k}^0) = 0$ *for some* $\mathbf{k}^0 \in [0,1]^N$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j0} = \mathbf{A}_0^{-1}\mathbf{A}_j, \quad j = 1, ..., N.$

Step 1. Define $\pi_1 = \det \mathbf{E}_1, \quad \mathbf{E}_1 = \mathbf{I} + \langle \mathbf{B}_1 \mathbf{A}_{10} \rangle_1.$

If $\pi_1(\mathbf{k}_1^0) = 0$ *for some* $\mathbf{k}_1^0 \in [0,1]^{N-1}$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j1} = \mathbf{A}_{j0} - \mathbf{A}_{10}\mathbf{E}_1^{-1}\langle \mathbf{B}_1 \mathbf{A}_{j0} \rangle_1, \quad j = 2, ..., N.$

Step 2. Define $\pi_2 = \det \mathbf{E}_2, \quad \mathbf{E}_2 = \mathbf{I} + \langle \mathbf{B}_2 \mathbf{A}_{21} \rangle_2.$

If $\pi_2(\mathbf{k}_2^0) = 0$ *for some* $\mathbf{k}_2^0 \in [0,1]^{N-2}$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j2} = \mathbf{A}_{j1} - \mathbf{A}_{21}\mathbf{E}_2^{-1}\langle \mathbf{B}_2 \mathbf{A}_{j1} \rangle_2, \quad j = 3, ..., N.$

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Step N. Define $\pi_N = \det \mathbf{E}_N, \quad \mathbf{E}_N = \mathbf{I} + \langle \mathbf{B}_N \mathbf{A}_{N,N-1} \rangle_N$. *If* $\pi_N = 0$ *then* $\mathcal{A}$ *is non-invertible else* $\mathcal{A}$ *is invertible.*

Summary

The following expansion as a product of elementary operators is fulfilled

$$\mathcal{A} = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1 + ... + \mathbf{A}_N \langle \mathbf{B}_N \cdot \rangle_N$$

$$= (\mathbf{A}_0 \cdot)(\mathbf{I} + \mathbf{A}_{10} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{I} + \mathbf{A}_{21} \langle \mathbf{B}_2 \cdot \rangle_2)...(\mathbf{I} + \mathbf{A}_{N,N-1} \langle \mathbf{B}_N \cdot \rangle_N),$$

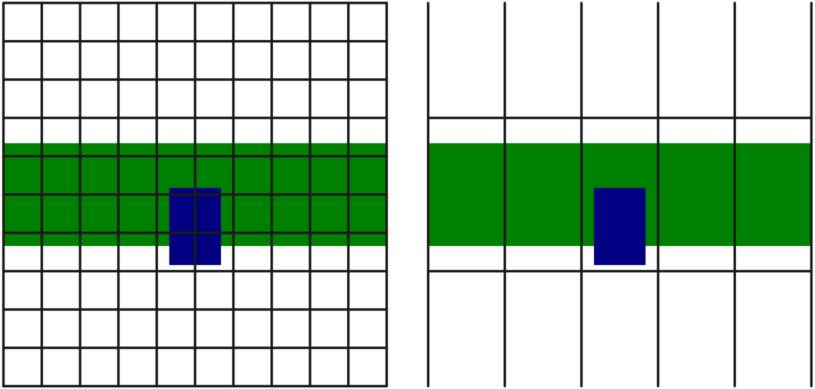
where $\mathbf{A}_{ij}$ are derived from $\mathbf{A}_n$, $\mathbf{B}_n$ by using algebraic operations (including taking inverse matrices) and a few number of integrations. The inverse is

$$\mathcal{A}^{-1} = (\mathbf{I} - \mathbf{A}_{N,N-1} \mathbf{E}_N^{-1} \langle \mathbf{B}_N \cdot \rangle_N)...(\mathbf{I} - \mathbf{A}_{10} \mathbf{E}_1^{-1} \langle \mathbf{B}_1 \cdot \rangle_1)(\mathbf{A}_0^{-1} \cdot),$$

where $\mathbf{E}_j = \mathbf{I} + \langle \mathbf{B}_j \mathbf{A}_{j,j-1} \rangle_j$. The determinant is

$$\boldsymbol{\pi}(\mathcal{A}) = (\pi_1, ..., \pi_N), \quad \pi_j = \det \mathbf{E}_j.$$

Embedded defects



Determinants in the case of embedded defects

In this case the operator has a form

$$\mathcal{A}\cdot = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \cdot \rangle_1 + ... + \mathbf{A}_N \langle \cdot \rangle_N,$$

where $\mathbf{A}_n$ does not depend on $k_1, ..., k_n$.
Define the matrix-valued integral continued fractions

$$\mathbf{C}_0 = \mathbf{A}_0, \quad \mathbf{C}_1 = \mathbf{A}_1 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_0} \right\rangle_1^{-1}, \quad \mathbf{C}_2 = \mathbf{A}_2 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_1 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_0} \right\rangle_1^{-1}} \right\rangle_2^{-1}$$

and so on $\mathbf{C}_j = \mathbf{A}_j + \langle \mathbf{C}_{j-1}^{-1} \rangle_j^{-1}$. Then

$$\pi_j(\mathcal{A}) = \det(\langle \mathbf{C}_{j-1}^{-1} \rangle_j \mathbf{C}_j).$$

Note that if all $\mathbf{A}_j$ are self-adjoint then $\mathcal{A}$ is self-adjoint and all $\mathbf{C}_j$ are self-adjoint. J. Math. Phys., 2017

The spectrum of $\mathcal{A}$ has the form

$$\mathrm{sp}(\mathcal{A}) = \bigcup_{n=0}^{N} \sigma_n, \quad \sigma_n = \{\lambda : \ \widetilde{\pi}_n = 0 \textit{ for some } \mathbf{k}\},$$

$$\textit{where} \quad \widetilde{\pi}_n \equiv \pi_n(\mathcal{A} - \lambda\mathcal{I}) \equiv \pi_n(\lambda, k_{n+1}, ..., k_N).$$

The component σ_0 coincides with the spectrum of purely periodic operator $\mathbf{A}_0\mathbf{u}$ without defects. All components σ_n, $n < N$ are continuous (intervals), the component σ_N is discrete.
Also note that σ_n does not depend on the defects of dimensions greater than n, i.e. of $\mathbf{A}_{n+1}$, $\mathbf{B}_{n+1}$, $\mathbf{A}_{n+2}$, $\mathbf{B}_{n+2}$ and so on.

Determinants of periodic operators with defects

For all continuous matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ on $[0,1]^N$ of appropriate sizes introduce

$$\mathfrak{H} = \{\mathcal{A} : \ \mathcal{A} = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1 + ... + \mathbf{A}_N \langle \mathbf{B}_N \cdot \rangle_N\} \subset \mathcal{B}(L^2_{N,M}),$$

$$\mathfrak{G} = \{\mathcal{A} \in \mathfrak{H} : \ \mathcal{A} \textit{ is invertible}\}.$$

Theorem (arxiv.org, 2015)

The set $\mathfrak{H}$ is a an operator algebra. The subset $\mathfrak{G}$ is a group. The mapping

$$\boldsymbol{\pi}(\mathcal{A}) := (\pi_0(\mathcal{A}), ..., \pi_N(\mathcal{A}))$$

is a group homomorphism between $\mathfrak{G}$ and $\mathcal{C}_0 \times \mathcal{C}_1 \times ... \times \mathcal{C}_N$, where $\mathcal{C}_n$ is the commutative group of non-zero continuous functions depending on $(k_{n+1}, ..., k_N) \in [0,1]^{N-n}$.

Traces of periodic operators with defects

Define

$$\boldsymbol{\tau}(\mathcal{A}) = \lim_{t\to 0} \frac{\boldsymbol{\pi}(\mathcal{I} + t\mathcal{A}) - \boldsymbol{\pi}(\mathcal{I})}{t}.$$

Then

Theorem (arxiv.org, 2015)

The following identities are fulfilled

$$\boldsymbol{\tau}(\mathcal{A}) = (\operatorname{Tr} \mathbf{A}_0, \langle \operatorname{Tr} \mathbf{B}_1 \mathbf{A}_1 \rangle_1, ..., \langle \operatorname{Tr} \mathbf{B}_N \mathbf{A}_N \rangle_N),$$

$$\boldsymbol{\tau}(\alpha\mathcal{A} + \beta\mathcal{B}) = \alpha\boldsymbol{\tau}(\mathcal{A}) + \beta\boldsymbol{\tau}(\mathcal{B}), \quad \boldsymbol{\tau}(\mathcal{A}\mathcal{B}) = \boldsymbol{\tau}(\mathcal{B}\mathcal{A}),$$

$$\boldsymbol{\pi}(e^{\mathcal{A}}) = e^{\boldsymbol{\tau}(\mathcal{A})}, \quad \boldsymbol{\pi}(\mathcal{A}\mathcal{B}) = \boldsymbol{\pi}(\mathcal{A})\boldsymbol{\pi}(\mathcal{B}).$$

Example. Laplace operator.

Continuous Laplace operator

$$\Delta U(\mathbf{x}) = \sum_{n=1}^{N} \frac{\partial^2}{\partial x_j^2} U(\mathbf{x}), \quad \mathbf{x} = (x_n) \in \mathbb{R}^N.$$

Discrete approximation of Laplace operator with a step $h \in \mathbb{R}$ is

$$\Delta_{\mathrm{discr}} U(h\mathbf{n}) = \sum_{n=1}^{N} \frac{U(h\mathbf{n} + h\mathbf{e}_n) - 2U(h\mathbf{n}) + U(h\mathbf{n} - h\mathbf{e}_n)}{h^2},$$

where

$$\mathbf{n} \in \mathbb{Z}^N, \quad \mathbf{e}_n = (\delta_{mn})_{m=1}^{N} \textit{ is a basis},$$

or easier

$$\Delta_{\mathrm{discr}} U_{\mathbf{n}} = \sum_{\mathbf{n}' \sim \mathbf{n}} (U_{\mathbf{n}'} - U_{\mathbf{n}}), \quad \mathbf{n} \in \mathbb{Z}^N,$$

where $\mathbf{n}' \sim \mathbf{n}$ means neighbor points.

Example. Laplace operator.

To discrete Laplace operator

$$\Delta_{\mathrm{discr}} U_{\mathbf{n}} = \sum_{\mathbf{n}' \sim \mathbf{n}} (U_{\mathbf{n}'} - U_{\mathbf{n}}), \quad \mathbf{n} \in \mathbb{Z}^N.$$

we apply FFB transformation

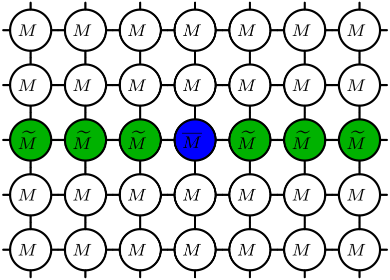
$$u(\mathbf{k}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} e^{2\pi i \mathbf{k}\cdot\mathbf{n}} U_{\mathbf{n}}, \quad \mathbf{k} = (k_n) \in [0,1]^N.$$

Then we obtain

$$\hat{\Delta}_{\mathrm{discr}} u(\mathbf{k}) = \sum_{\sigma = \pm 1, n = 1, .., N} (e^{2\pi i \sigma k_n} u(\mathbf{k}) - u(\mathbf{k}))$$

$$= \sum_{n=1}^{N} (e^{2\pi i k_n} + e^{-2\pi i k_n} - 2) u(\mathbf{k}) = \left(-4 \sum_{n=1}^{N} \sin^2 \pi k_n \right) u(\mathbf{k}).$$

Example. Uniform lattice with guide and single defect.



Wave equation has the form ($\lambda \sim \omega^2$ is an "energy")

$$-(\Delta_{\mathrm{disc}})u(x,y) = \lambda \begin{cases} \overline{M}u(x,y), & x = y = 0, \\ \widetilde{M}u(x,y), & x = 0, \ y \neq 0, \\ Mu(x,y), & \textit{otherwise} \end{cases} \qquad u \in \ell^2(\mathbb{Z}^2)$$

After applying Floquet-Bloch transformation it becomes

$$4(\sin^2 \pi k_1 + \sin^2 \pi k_2)\hat{u} = \lambda M \hat{u} + \lambda(\widetilde{M} - M) \int_0^1 \hat{u}\, dk_1 + \lambda(\overline{M} - \widetilde{M}) \int_0^1 \int_0^1 \hat{u}\, dk_1\, dk_2$$

Example. Uniform lattice with guide and single defect.

Taking $M = 1$ (uniform value) and denoting

$$A = 4(\sin^2 \pi k_1 + \sin^2 \pi k_2), \quad M_1 = \widetilde{M} - M, \quad M_2 = \overline{M} - \widetilde{M}$$

we may rewrite our wave equation as

$$(A - \lambda)u - \lambda M_1 \langle u \rangle_1 - \lambda M_2 \langle u \rangle_2 = 0 \quad \textit{or} \quad \mathcal{A}u = 0.$$

The determinant of $\mathcal{A}$ is $\boldsymbol{\pi}(\mathcal{A}) = (\pi_0, \pi_1, \pi_2)$. Using previously defined integral continued fractions (p.21) we can compute

$$\pi_0 = A - \lambda, \quad \pi_1 = 1 - \left\langle \frac{\lambda M_1}{\pi_0} \right\rangle_1, \quad \pi_2 = 1 - \left\langle \frac{\lambda M_2}{\pi_0 \pi_1} \right\rangle_2.$$

Thus the procedure of finding determinants consists of the steps "take inverse and integrate, take inverse and integrate...".

Example. Uniform lattice with guide and single defect.

Propagative dispersion curve is

$$\pi_0 = 0 \ \Leftrightarrow \ \lambda = \lambda_{\mathrm{p}}(k_1, k_2) = 4(\sin^2 \pi k_1 + \sin^2 \pi k_2).$$

Guided dispersion curve is

$$\pi_1 = 0 \ \Leftrightarrow \ \lambda = \lambda_{\mathrm{g}}(k_2) = \frac{-4\sin^2 \pi k_2 - 2 \pm 2\sqrt{4M_1^2 \sin^2 \pi k_2 (1 + \sin^2 \pi k_2) + 1}}{M_1^2 - 1}.$$

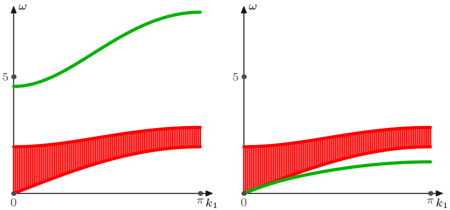
Localised eigenvalues $\lambda = \lambda_{\mathrm{loc}}$ are determined from the equation

$$\pi_2 = 0 \ \Leftrightarrow \ 1 + \int_0^1 \frac{\lambda M_2 dk_1}{\lambda M_1 + \sqrt{(\lambda - 2 - 4\sin^2 \pi k_1)^2 - 4}} = 0.$$

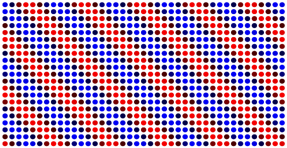
The total spectrum is

$$\sigma = \sigma_0 \cup \sigma_1 \cup \sigma_2, \quad \sigma_0 = \lambda_{\mathrm{p}}([0,1]^2), \quad \sigma_1 = \lambda_{\mathrm{g}}([0,1]), \quad \sigma_3 = \{\lambda_{\mathrm{loc}}\}.$$

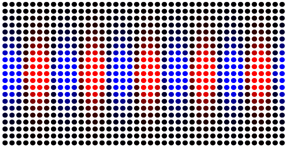
Example. Guided and localised spectrum.



propagative wave



guided wave

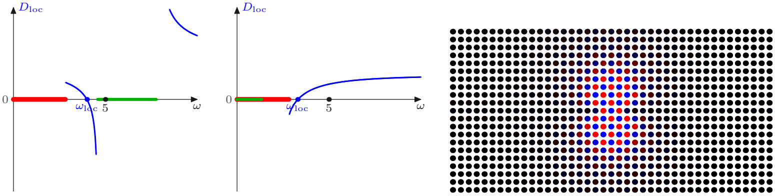


Propagative dispersion surface (red) is

$$\lambda_{\mathrm{p}}(k_1, k_2) = 4(\sin^2 \pi k_1 + \sin^2 \pi k_2).$$

Guided dispersion curve (green) is

$$\lambda_{\mathrm{g}}(k_2) = \frac{-4\sin^2 \pi k_2 - 2 \pm 2\sqrt{4M_1^2(1+\sin^2 \pi k_2)\sin^2 \pi k_2 + 1}}{M_1^2 - 1}.$$



Localised eigenvalues $\lambda = \lambda_{\mathrm{loc}}$ are determined from the equation

$$D_{\mathrm{loc}}(\lambda) := 1 + \int_0^1 \frac{\lambda M_2 \, dk_1}{\lambda M_1 + \sqrt{(\lambda - 2 - 4\sin^2 \pi k_1)^2 - 4}} = 0.$$

Example. Large mass of the guide

$$D_{\rm loc}(8) = 1 + \int_0^1 \frac{2M_2\,dk_1}{2M_1 + |\cos\pi k_1|\sqrt{2-\sin^2\pi k_1}} =$$

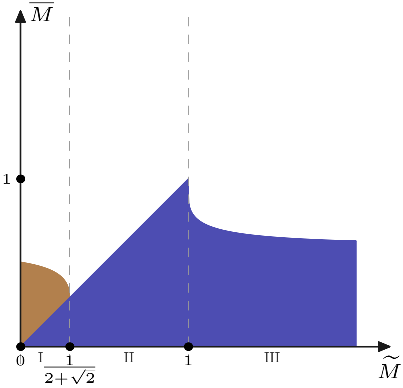
$$1 + \frac{M_2}{M_1}\left(1 - \int_0^{1/2} \frac{\cos\pi k_1\sqrt{2-\sin^2\pi k_1}\,dk_1}{M_1} + O\left(\frac{1}{M_1^2}\right)\right) =$$

$$1 + \frac{M_2}{M_1}\left(1 - \left(\frac{1}{4} + \frac{1}{2\pi}\right)\frac{1}{M_1} + O\left(\frac{1}{M_1^2}\right)\right)$$

Since $D_{\rm loc}(+\infty) = \frac{1+M_1+M_2}{1+M_1} = \frac{\overline{M}}{\widetilde{M}} > 0$ and $D_{\rm loc}$ is monotonic for $\lambda > 8$ then it has zero (which is an isolated eigenvalue) if and only if $D_{\rm loc}(8) < 0$ which yields

$$M_2 = -M_1 - \frac{1}{4} - \frac{1}{2\pi} + O\left(\frac{1}{M_1}\right) \;\Rightarrow\; \overline{M} = \frac{3}{4} - \frac{1}{2\pi} + O\left(\frac{1}{\widetilde{M}}\right).$$

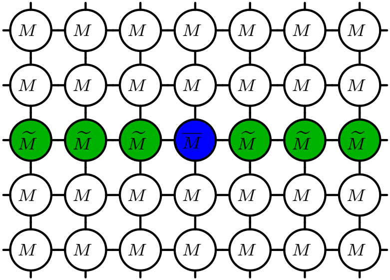
Example. Masses for which the local eigenvalue exists.

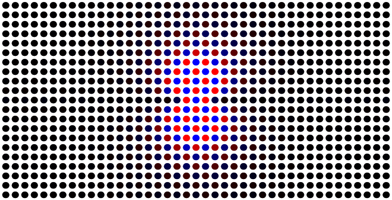


$$\text{upper limit of } \overline{M} = 1 + \frac{\pi}{4\ln(\widetilde{M}-1)} + \dots, \quad \widetilde{M} \to 1+0,$$

$$\text{upper limit of } \overline{M} \to \frac{3}{4} - \frac{1}{2\pi}, \quad \widetilde{M} \to \infty.$$

Example. Uniform lattice with guide and single defect.





For the random uniform distribution of the masses of the media, of the guide, and of the point defect ($< M$) the probability of existence of the isolated eigenvalue is exactly

$$\frac{3}{4} - \frac{1}{2\pi}.$$

Comput. Mech., 2014

Wave propagation in the lattice with defects and sources

Wave equation

$$\Delta_{\mathrm{discr}} U_{\mathbf{n}}(t) = S_{\mathbf{n}}^2 \ddot{U}_{\mathbf{n}}(t) + \sum_{\mathbf{n}' \in \mathcal{N}_{\mathrm{F}}} F_{\mathbf{n}'}(t) \delta_{\mathbf{n}\mathbf{n}'}, \quad \mathbf{n} \in \mathbb{Z}^2$$

Assuming harmonic sources and applying F-B transformation we obtain

$$Av = -\omega^2 \mathbf{a}^* \mathbf{S} \langle v\mathbf{a} \rangle + \mathbf{b}^* \mathbf{f}.$$

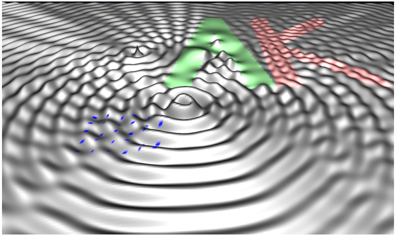
Using the explicit form for inverse integral operator (p.19) we may derive explicit solution of the last equation

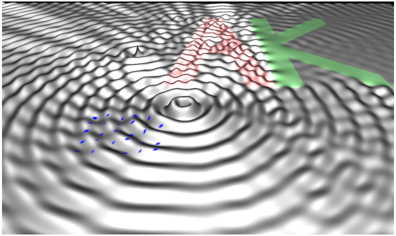
$$v = A^{-1} \left(-\omega^2 \mathbf{a}^* \mathbf{S} \mathbf{G} \left\langle \frac{\mathbf{a}\mathbf{b}^*}{A} \right\rangle + \mathbf{b}^* \right) \mathbf{f},$$

where

$$\mathbf{G} = (\mathbf{I} + \omega^2 \mathbf{A} \mathbf{S})^{-1}, \quad \mathbf{A} = \left\langle \frac{\mathbf{a}\mathbf{a}^*}{A} \right\rangle.$$

Wave simulations. Inverse problem.





Two formulas allow us to recover the defect properties from the information about amplitudes of waves at the receivers

$$\mathbf{S}\langle u\mathbf{a}\rangle = \omega^{-2}\mathbf{C}^{-1}\left(\left\langle \frac{\mathbf{c}\mathbf{b}^*}{A}\right\rangle \mathbf{f} - \langle u\mathbf{c}\rangle\right),$$

$$\langle u\mathbf{a}\rangle = -\mathbf{A}\mathbf{C}^{-1}\left(\left\langle \frac{\mathbf{c}\mathbf{b}^*}{A}\right\rangle \mathbf{f} - \langle u\mathbf{c}\rangle\right) + \left\langle \frac{\mathbf{a}\mathbf{b}^*}{A}\right\rangle \mathbf{f},$$

where

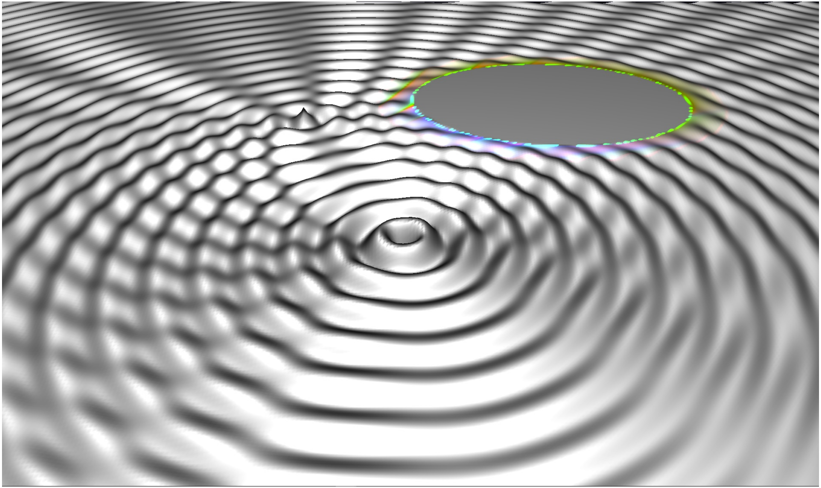
$$\mathbf{c} = \begin{pmatrix} e^{-i\mathbf{n}_1\cdot\mathbf{k}} \\ \dots \\ e^{-i\mathbf{n}_N\cdot\mathbf{k}} \end{pmatrix}_{\mathbf{n}_j\in\mathcal{N}_{\mathrm{R}}}, \quad \mathbf{C} = \left\langle \frac{\mathbf{c}\mathbf{a}^*}{A}\right\rangle.$$

Eur. J. Mech. A-Solid., 2015

Inverse Probl., 2016

Cloaking device.

The same formulas are applicable for constructing "invisible objects"



Remark about extended Fredholm operators

We have seen that for the operators

$$\mathcal{A} = \mathbf{A}_0 \cdot + \mathbf{A}_1 \int_0^1 \mathbf{B}_1 \cdot dk_1 + ... + \mathbf{A}_N \int_0^1 .. \int_0^1 \mathbf{B}_N \cdot dk_1..dk_N,$$

where $\mathbf{A}_j \equiv \mathbf{A}_j(\mathbf{k}),\ \ \mathbf{B}_j \equiv \mathbf{B}_j(\mathbf{k})$ *are* $M \times M$ *matrices and* $\mathbf{k} = (k_1, ..., k_N)$

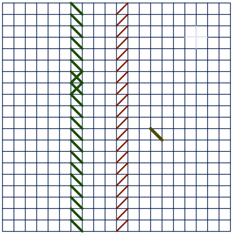
the procedure for finding inverse operators, spectra, etc is based on some matrix operations and few number of integrations. We call such procedures "explicit". By the continuity, these procedures can be extended to the case

$$\mathcal{A} = \mathbf{A}_0 \cdot + \int_0^1 \mathbf{A}_1 \cdot dx_1 + ... + \int_0^1 .. \int_0^1 \mathbf{A}_N \cdot dx_1..dx_N,$$

where $\mathbf{A}_j \equiv \mathbf{A}_j(\mathbf{k}, \mathbf{x}_j)$ *and* $\mathbf{x}_j = (x_1, ..., x_j), \quad \cdot = u(\mathbf{k}_{N-j}, \mathbf{x}_j).$

Of course, we lose some kind of explicitness in this case.

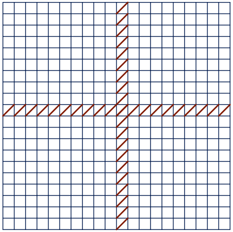
Remark about perpendicular defects, 2D case



Consider 2D case. Algebras of parallel defects are

$$\mathfrak{H} = \{\mathbf{A}_0 \cdot + \mathbf{A}_1 \int_0^1 \mathbf{B}_1 \cdot dk_1 + \mathbf{A}_2 \int_0^1 \int_0^1 \mathbf{B}_2 \cdot dk_1 dk_2\},$$

$$\widetilde{\mathfrak{H}} = \{\mathbf{A}_0 \cdot + \mathbf{A}_1 \int_0^1 \mathbf{B}_1 \cdot dk_2 + \mathbf{A}_2 \int_0^1 \int_0^1 \mathbf{B}_2 \cdot dk_1 dk_2\}$$



Algebra of perpendicular defects is

$$\mathfrak{A} = \{\mathbf{A}_0 \cdot + \mathbf{A}_1 \int_0^1 \mathbf{B}_1 \cdot dk_1 + \mathbf{A}_2 \int_0^1 \mathbf{B}_2 \cdot dk_2 + \int_0^1 \int_0^1 \mathbf{B}_3 \cdot dx_1 dx_2\}.$$

Even for "simple" operators from $\mathfrak{A}$ we lose the "explicitness" of finding inverse operators and spectra.

J. Math. Anal. Appl., 2016

Finite-dimensional approximation, 2D case

Suppose that all kernels are p-step (piecewise constant) functions. Then

$$\mathfrak{A} = \mathrm{Alg}\left(\chi_i(k_1)\cdot,\ \chi_j(k_2)\cdot,\ \int_0^1 \cdot dk_1,\ \int_0^1 \cdot dk_2\right),\ \chi_i(k) = \begin{cases} 1, & k \in [\frac{i-1}{p}, \frac{i}{p}), \\ 0, & otherwise. \end{cases}$$

and all operators from $\mathfrak{A}$ have a form

$$\mathcal{A} = A \cdot + \int_0^1 B \cdot dx_1 + \int_0^1 C \cdot dx_2 + \int_0^1 \int_0^1 D \cdot dx_1 dx_2,\ where$$

$$A(k_1, k_2) = \sum_{i,j=1}^{p} a_{ij} \chi_i(k_1)\chi_j(k_2),\quad B(k_1, k_2, x_1) = \sum_{i,j,m=1}^{p} b_{ijm} p \chi_i(k_1)\chi_j(k_2)\chi_m(x_1),$$

$$C = \sum_{i,j,n=1}^{p} c_{ijn} p \chi_i(k_1)\chi_j(k_2)\chi_n(x_2),\quad D = \sum_{i,j,m,n=1}^{p} d_{ijmn} p^2 \chi_i(k_1)\chi_j(k_2)\chi_m(x_1)\chi_n(x_2).$$

and all coefficients $a_{ij}, b_{ijm}, c_{ijn}, d_{ijmn} \in \mathbb{C}$.

Expansion as a product of simple algebras

Introduce the following mapping

$$\sigma : \mathfrak{A} \to \mathbb{C}^{p^2} \times (\mathbb{C}^{p\times p})^p \times (\mathbb{C}^{p\times p})^p \times \mathbb{C}^{p^2\times p^2}, \quad \textit{where}$$

$$\sigma = ((\sigma^1_{ij})^p_{i,j=1}, (\sigma^2_j)^p_{j=1}, (\sigma^3_i)^p_{i=1}, \sigma^4),$$

and matrices $\sigma^1_{ij}, \sigma^2_j, \sigma^3_i, \sigma^4$ are defined by

$$\sigma^1_{ij}(\mathcal{A}) = a_{ij} \in \mathbb{C},$$

$$\sigma^2_j(\mathcal{A}) = (\delta_{im} a_{ij} + b_{ijm})^p_{i,m=1} \in \mathbb{C}^{p\times p},$$

$$\sigma^3_i(\mathcal{A}) = (\delta_{jn} a_{ij} + c_{ijn})^p_{j,n=1} \in \mathbb{C}^{p\times p},$$

$$\sigma^4(\mathcal{A}) = (\delta_{im}\delta_{jn} a_{ij} + \delta_{jn} b_{ijm} + \delta_{im} c_{ijn} + d_{ijmn})^{p^2}_{r,s=1} \in \mathbb{C}^{p^2\times p^2},$$

where $r = i + p(j-1),\ s = m + p(n-1)$ *and* δ *is the Kronecker* δ.

Theorem (2016)

1) The mapping σ is an algebra isomorphism. 2) The operator $\mathcal{A}$ is invertible if all matrices $\sigma(\mathcal{A})$ are invertible and

$$\mathcal{A}^{-1} = \sigma^{-1}((\sigma^1_{ij})^{-1}, (\sigma^2_j)^{-1}, (\sigma^3_i)^{-1}, (\sigma^4)^{-1}).$$

3) The spectrum of $\mathcal{A}$ consists of all eigenvalues of matrices $\sigma(\mathcal{A})$.

Example 1.

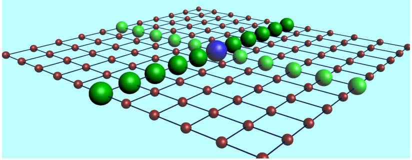
Consider the simplest case $p = 1$. Then

$$\left(a \cdot + b \int_0^1 \cdot dk_1 + c \int_0^1 \cdot dk_2 + d \int_0^1 \int_0^1 \cdot dk_1 dk_2 \right)^{-1} =$$

$$a^{-1} \cdot - \frac{b}{a(a+b)} \int_0^1 \cdot dk_1 - \frac{c}{a(a+c)} \int_0^1 \cdot dk_2 +$$

$$\frac{(2a+b+c+d)bc - a^2 d}{a(a+b)(a+c)(a+b+c+d)} \int_0^1 \int_0^1 \cdot dk_1 dk_2.$$

Example 2. Schrödinger operator.



Consider a spectral problem for the Schrödinger operator

$$\mathcal{A}:\ell^2(\mathbb{Z}^2)\to\ell^2(\mathbb{Z}^2),\quad \mathcal{A}U_{\mathbf{n}}=-\Delta U_{\mathbf{n}}+V_{\mathbf{n}}U_{\mathbf{n}},\quad \mathbf{n}\in\mathbb{Z}^2,$$

$$V_{\mathbf{n}}=\begin{cases}0, & n_1n_2\neq 0,\\ V_1, & n_1=0,\, n_2\neq 0,\\ V_2, & n_2=0,\, n_1\neq 0\\ V_1+V_2+V_3, & n_1=n_2=0\end{cases},\quad \mathbf{n}=(n_1,n_2)\in\mathbb{Z}^2.$$

After applying FFB transformation $\mathcal{F}$ it takes the form

$$\hat{\mathcal{A}}=\mathcal{F}\mathcal{A}\mathcal{F}^{-1}:L^2\to L^2,\quad \hat{\mathcal{A}}=A\cdot+V_1\int_0^1\cdot dx_1+V_2\int_0^1\cdot dx_2+V_3\int_0^1\int_0^1\cdot dx_1dx_2,$$

where $A=4-2\cos 2\pi k_1-2\cos 2\pi k_2$.

Example 2. Discrete approximation of the operator

Following the notations before the theorem we have

$$a_{ij} = A((i-1/2)/p,(j-1/2)/p), \quad b_{ijm} = V_1/p, \quad c_{ijn} = V_2/p, \quad d_{ijmn} = V_3/p^2$$

for $i,j = 1,...,p$. Recall that the matrices σ are defined by

$$\sigma^1_{ij} = a_{ij} \in \mathbb{C},$$

$$\sigma^2_j = (\delta_{im} a_{ij} + b_{ijm})^p_{i,m=1} \in \mathbb{C}^{p\times p},$$

$$\sigma^3_i = (\delta_{jn} a_{ij} + c_{ijn})^p_{j,n=1} \in \mathbb{C}^{p\times p},$$

$$\sigma^4 = (\delta_{im}\delta_{jn} a_{ij} + \delta_{jn} b_{ijm} + \delta_{im} c_{ijn} + d_{ijmn})^{p^2}_{r,s=1} \in \mathbb{C}^{p^2\times p^2}.$$

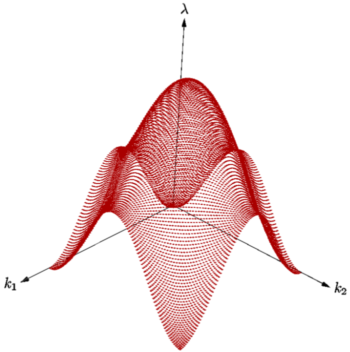
The difference ε_p between the initial operator and the approximated one (and, hence, the distance between spectra) has the form

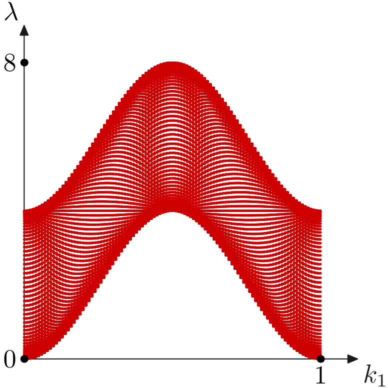
$$\mathrm{dist}(\mathrm{sp}(\hat{\mathcal{A}}), \mathrm{sp}(\hat{\mathcal{A}}_p)) \leqslant \varepsilon_p = \|\hat{\mathcal{A}} - \hat{\mathcal{A}}_p\| \leqslant \frac{1}{2p} \max_{k_1,k_2} |\nabla A(k_1,k_2)| \leqslant \frac{4\pi}{p}.$$

We consider the potentials

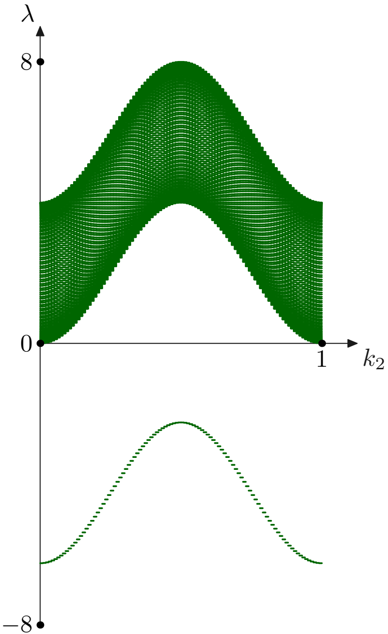
$$V_1 = -8, \quad V_2 = 2, \quad V_3 = 1, \quad p = 100.$$

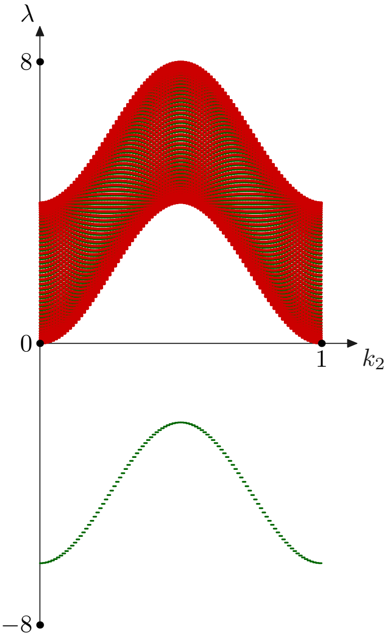
Example 2. Propagative spectrum: eig. of σ^1_{ij}



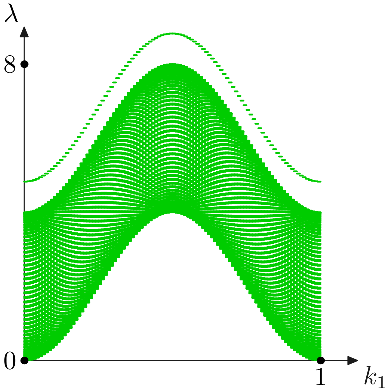


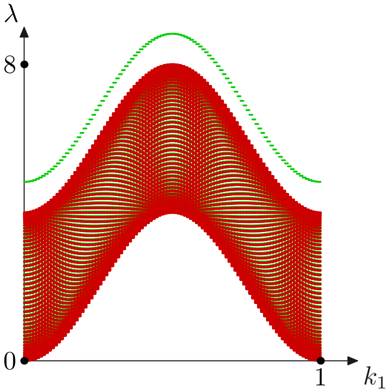
Example 2. Guided spectrum: eig. of σ_j^2



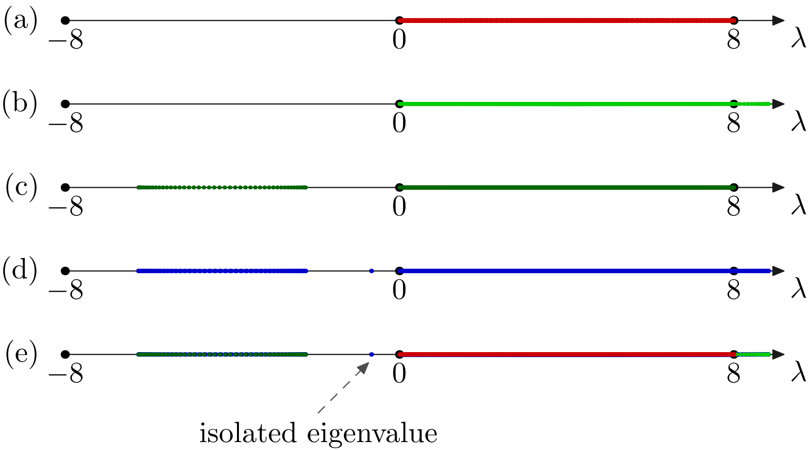


Example 2. Guided spectrum: eig. of σ_i^3





Example 2. Local spectrum: eig. of σ^4



General matrix-valued multidimensional case

Consider the general algebra

$$\mathfrak{L}_{N,M}^{\frac{1}{p}} = \mathrm{Alg}\left(\mathbf{A}\cdot, \chi_i(k_j)\cdot, \int_0^1 \cdot dx_j\right),$$

where $\mathbf{A} \in \mathbb{C}^{M\times M}$ are all matrices of the dimension M, and $i = 1, .., p$, $j = 1, .., N$. Then it can be shown that

$$\mathfrak{L}_{N,M}^{\frac{1}{p}} \simeq \prod_{n=0}^{N} (\mathbb{C}^{Mp^n \times Mp^n})^{\binom{N}{n} p^{N-n}},$$

where $\binom{N}{n}$ are binomial coefficients. The isomorphisms σ and σ^{-1} have explicit forms.