Spectral gap for graphene quantum dots

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Differential Operators on Graphs and Waveguides
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Outline

1. Dirac operator with ∞-mass boundary conditions
2. Upper bounds on the size of the spectral gap
3. Key ideas of the proof

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The off-diagonal entries resemble Cauchy-Riemann operators: interplay with complex analysis.

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$$\dots \le -\mu_2(\Omega) \le -\mu_1(\Omega) < 0 < \mu_1(\Omega) \le \mu_2(\Omega) \le \dots$$

$\mu_1(\Omega) := \inf(\sigma(\mathsf{D}_\Omega) \cap \mathbb{R}_+)$ describes the size of the spectral gap.

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$$\mathsf{D}_{\mathbb{R}^2} + \underbrace{M\sigma_3 \chi_{\mathbb{R}^2 \setminus \Omega}}_{\text{mass term}} \xrightarrow{M \to \infty} \mathsf{D}_\Omega, \qquad \text{(in a proper sense).}$$

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Eigenmodes of such GQD are effectively described by D_Ω.

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Aim: to get a bound on $\mu_1(\Omega)$ in the spirit of an isoperimetric inequality.

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Proposition

$\mu_1(\mathbb{D}) > 0$ is the smallest non-negative solution of $J_0(\mu) = J_1(\mu)$. An eigenfunction associated to $\mu_1(\mathbb{D})$ is

$$u_\circ(r,\theta) := \begin{pmatrix} J_0(\mu_1(\mathbb{D})r) \\ \mathrm{i}e^{\mathrm{i}\theta} J_1(\mu_1(\mathbb{D})r) \end{pmatrix}.$$

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- ✗ Symmetric decreasing rearrangement.
- ✗ Parallel coordinates.
- ✓ Conformal maps.
- ✓ Shrinking coordinates.

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Geometric estimates of $\|f'\|_{\mathcal{H}^2(\mathbb{D})}$

Star-shaped (nearly circular): Warschawski-50, Specht-51, Gaier-62

Convex: Kovalev-2017

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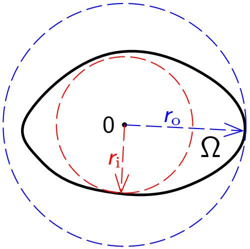

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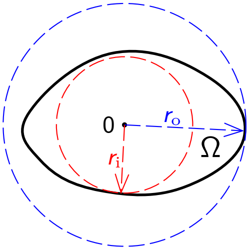

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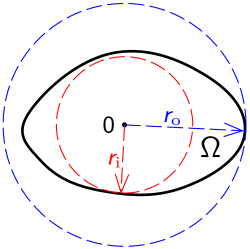

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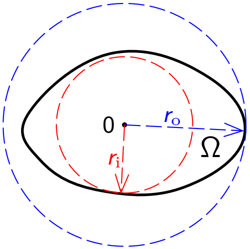

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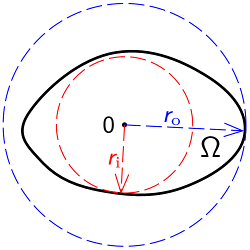

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Inspired by $\lambda_1^{\mathrm{Dir}}(\Omega) \leq \frac{|\partial\Omega|}{2r_i|\Omega|}\lambda_1^{\mathrm{Dir}}(\mathbb{D})$ for convex Ω's (Pólya-Szegő-51)

Discussion

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$$\mathcal{F}_{\mathrm{K}}(\Omega) = \mathcal{F}_{\mathrm{K}}(\mathbb{D}_r) = \frac{1}{r} \qquad \Longrightarrow \qquad \mu_1(\Omega) < \mu_1(\mathbb{D}_r).$$

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Private communication with L. Kovalev

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Definition (Nearly circular domains)

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Outline

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The disk or not the disk?

$$\begin{cases} |\Omega| = \pi \\ \Omega \ncong \mathbb{D} \end{cases} \quad \overset{??}{\Longrightarrow} \quad \mu_1(\Omega) > \mu_1(\mathbb{D}).$$

Thank you

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V. L. and T. Oumières-Bonafos, A sharp upper bound on the spectral gap for convex graphene quantum dots, arXiv:1812.03029.

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Thank you for your attention!