Breathers in Nonlinear Klein-Gordon Equations on a Periodic Necklace Graph

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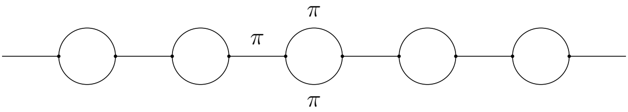
Breather solutions in nonlinear Klein-Gordon equations

Nonlinear *Klein-Gordon equation* on the necklace graph

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - (\alpha + \varepsilon) u(x,t) + u(x,t)^3, \quad x \in \operatorname{int} \Gamma,\ t \in \mathbb{R},$$

and Kirchhoff boundary conditions at the vertex points (continuity and conservation of flows).



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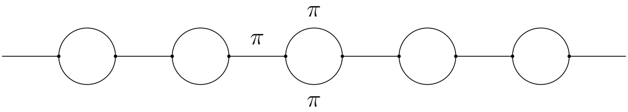


Breather solutions:

- real-valued,
- time-periodic, $u(t) = u\left(t + \frac{2\pi}{\omega}\right),\ t \in \mathbb{R}$,
- spatially localized, $\lim_{|x|\to\infty} |u(x,t)| = 0,\ t \in \mathbb{R}$.

Existence results on the real line

- Sine-Gordon equation

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + \sin(u(x,t)) = 0, \quad x,t \in \mathbb{R}.$$

has breathers, explicitly given by

$$u(x,t) = 4 \arctan\left(\frac{m \sin(\omega t)}{\omega \cosh(mx)}\right), \quad m^2 + \omega^2 = 1.$$

- No persistence under small perturbations:

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) + f(u) = 0, \quad f(0) = 0,\ f'(0) = 1,$$

has no breathers for $f(u) \neq \sin(u)$. [Denzler '93]

Breathers in nonlinear PDEs are very rare!

Existence results

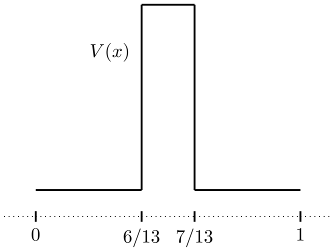
However, the situation is different if one introduces spatial inhomogeneities.

- Breathers on lattices [MacKay, Aubry '94]:

$$\partial_t^2 u_n(t) - (u_{n+1} - 2u_n + u_{n-1}) + f(u_n) = 0, \; f(0) = 0, \; f'(0) > 0.$$

- Small amplitude breathers in nonlinear Klein-Gordon equation with periodic coefficients [Blank, Chirilus-Bruckner, Lescarret, Schneider '11]:

$$V(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - q(x)u(x,t) + u(x,t)^3, \quad x, t \in \mathbb{R}.$$



$q(x) = (q_0 - \varepsilon^2)V(x)$

Breather solutions in nonlinear Klein-Gordon equations on the necklace graph

Existence theorem:

Let k be an odd integer. For sufficiently small $\varepsilon > 0$, the cubic Klein-Gordon equation

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - (k^2/4 + \varepsilon) u(x,t) + u(x,t)^3$$

with Kirchhoff boundary conditions at the vertices possesses breather solutions of amplitude $\mathcal{O}(\sqrt{\varepsilon})$ and frequency $\omega = k/2$. These solutions are symmetric in the upper and lower semicircle.

Precisely, there exist functions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

- $u(x,t) = u(x, t + \frac{2\pi}{\omega})$, for all $t, x \in \mathbb{R}$,
- $\lim_{|x|\to\infty} u(x,t) e^{\beta |x|} = 0$, for all $t \in \mathbb{R}$ and a constant $\beta > 0$.

Creation of standing pulse

- "time-periodic" $u(x,t) = u\left(x, t + \frac{2\pi}{\omega}\right)$, $x, t \in \mathbb{R}$, "real-valued".
  $\rightsquigarrow$ Fourier series representation in time

$$u(x,t) = \sum_{m \in \mathbb{N}_{\mathrm{odd}}} u_m(x) \cos(m\omega t)$$

gives system with new dynamic variable x ($\to$ **spatial dynamics**)

$$-m^2\omega^2 u_m = \partial_x^2 u_m - (\alpha + \varepsilon) u_m + (u * u * u)_m, \qquad m \in \mathbb{N}_{\mathrm{odd}}.$$

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- "spatially localized" $\lim_{|x| \to \infty} |u(x,t)| = 0$, $t \in \mathbb{R}$.
  $\rightsquigarrow$ Find a homoclinic to zero in this infinite dimensional system

Is center manifold reduction possible?

Ingredients of the proof

1. Spectral situation and time-2π-maps
2. Discrete center manifold theorem
3. Analysis of the reduced system

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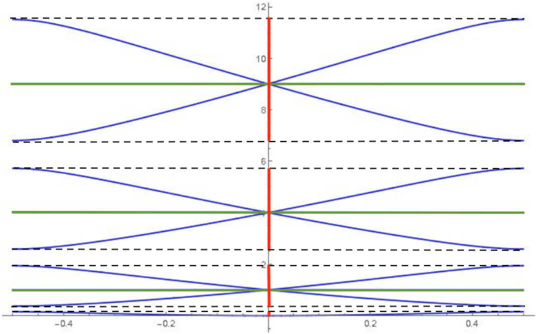
1) Spectral situation

1. $\sigma_{pp}(-\partial_x^2|_\Gamma) = \{m^2 \,:\, m \in \mathbb{N}\}$: eigenvalues of infinite multiplicity and antisymmetric eigenstates (generated by simple loop states).
2. $\sigma_{\mathrm{ac}}(-\partial_x^2|_\Gamma)$: band-gap structure with symmetric (generalized) eigenstates.

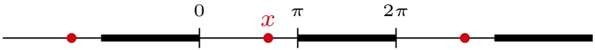
Floquet-Bloch bands:



Bloch wave ansatz $W(x) = e^{ilx} f(l,x)$ with 2π-periodic functions $f(l)$ leads to the eigenvalue problems $-(\partial_x + il)^2 f = \omega(l) f$ and $\sigma(-\partial_x^2) = \bigcup_{l\in[-1/2,1/2]} \sigma(-(\partial_x + il)^2)$.

1) Time-2π-maps on the invariant subspace of symmetric functions

We identify the symmetric necklace graph with the real line equipped with a singular periodic potential.



Family of **time-**2π**-maps** for $x \in [0, 2\pi)$ for symmetric solutions:

$$TU_n(x) = U_{n+1}(x), \quad n \in \mathbb{Z}, \qquad U_n(x) = \begin{pmatrix} u(x + 2\pi n) \\ u'^{+}(x + 2\pi n) \end{pmatrix}$$

with right-sided derivatives u'^{+} at the vertex points.

1) Time-2π-maps and monodromy matrices

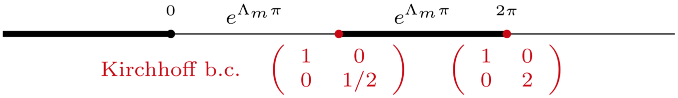
The spatial dynamics system

$$-m^2\omega^2 u_m = \partial_x^2 u_m - (\alpha+\varepsilon)u_m + (u*u*u)_m, \qquad m \in \mathbb{N}_{\mathrm{odd}}.$$

has time-2π-mappings

$$T(U_m)_n(x) = M_x(m^2\omega^2-\alpha)(U_m)_n(x) + N_m(x,\varepsilon,U_n),\ n\in\mathbb{Z},\ m\in\mathbb{N}_{\mathrm{odd}},$$

with explicitly computable monodromy matrices $M_x(m^2\omega^2-\alpha)$.



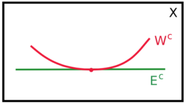
$\rightsquigarrow$ Idea: apply a **discrete version of the center manifold theorem** to the family of time-2π-maps

Ingredients of the proof

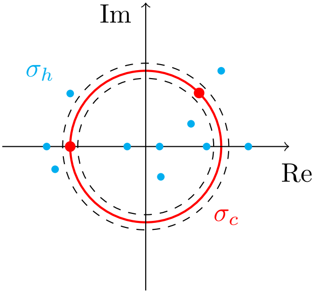
1. Spectral situation and time-2π-maps
2. Discrete center manifold theorem
3. Analysis of the reduced system

2) Discrete center manifold theorem

Center eigenvalues = eigenvalues on the unit circle.
To prove the existence of an (invariant) center manifold,



we need a **spectral gap** around the unit circle.



dimension of center manifold $= \#\{$eigenvalues on unit circle$\}$

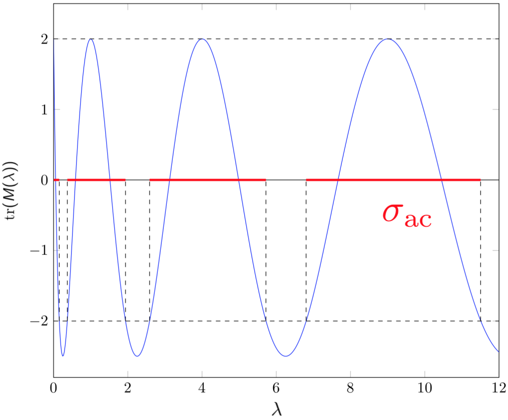
Center manifold theorem:
The dynamics of small solutions are determined on the center manifold.
$\rightarrow$ Flow can be restricted to the center manifold.

Reduction of dimension

2) Spectral situation on the necklace graph

Characterization of the spectrum with monodromy matrices:

$$\lambda \in \sigma_{\mathrm{ac}}(-\partial_x^2|_\Gamma) \Leftrightarrow |\mathrm{tr}(M(\lambda))| \leq 2$$
$$\Leftrightarrow \text{Eigenvalues of } M(\lambda) \text{ on unit circle}$$



2) Choice of breather frequency ω and constant α

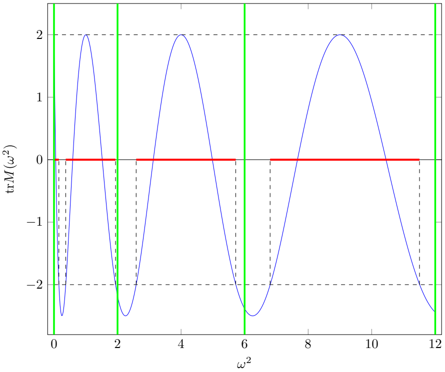
Key relation:

1. $\left|\mathrm{tr} M(m^2\omega^2 - \alpha)\right| \leq 2 \Leftrightarrow$ Eigenvalues of $M(m^2\omega^2 - \alpha)$ on unit circle
2. $\#\{$Eigenvalues of $M(m^2\omega^2 - \alpha)$ on unit circle$\}$ = dimension of center manifold

Choice of breather frequency:
For constants $\omega = k/2$, $\alpha = \omega^2$ with $k \in \mathbb{N}_{\mathrm{odd}}$ we find that

$$\left|\mathrm{tr} M(\omega^2 - \alpha)\right| = 2,$$
$$\left|\mathrm{tr} M(m^2\omega^2 - \alpha)\right| > 2,$$

for all odd $m \geq 3$.



2) Occurrence of the spectral gaps

Schrödinger operator with periodic potential on the real line:

$$H_{per} = -\frac{d^2}{dx^2} + V_{per} \quad \text{with}\, V_{per} \in C^m(\mathbb{R}),$$

has bandgap structure with width $\mathcal{O}(n^{-(m-1)})$ of the n^{th} spectral gap for $n \to \infty$. To avoid gaps becoming smaller, V_{per} might be at most once differentiable.

Linear part on necklace graphs gives rise to the desired gap structure

Ingredients of the proof

1. Spectral situation and time-2π-maps
2. Discrete center manifold theorem
3. Analysis of the reduced system

3) Analysis of the reduced system

Lowest order approximation of the dynamics on the center manifolds is given by the ODE

$$\partial_x^2 u_1(x) = \varepsilon^2 u_1(x) - u_1^3(x).$$

with $u_m = 0$ for all $m \geq 3$.

$\Rightarrow$ Existence of a homoclinic to zero [Pelinovsky, Schneider '16]

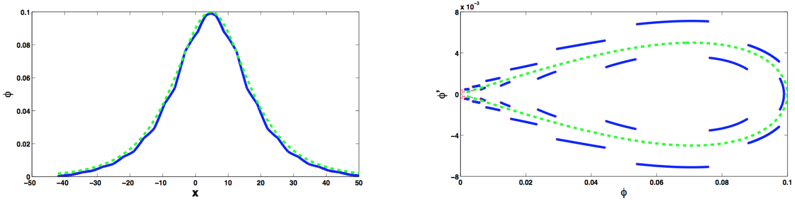


Figure: blue line: homoclinic on the graph;
dashed green line: homoclinic on the real line without potential

Persistence of the homoclinic under higher order perturbations:

- transversal intersection
- reversibility

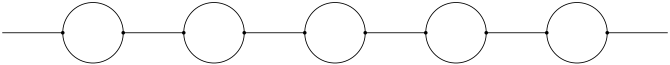
$\rightarrow$ phase portrait is reflection symmetric at the u_1-axis

Summary

Nonlinear Klein-Gordon equation on the necklace graph

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - (k^2/4 + \varepsilon) u(x,t) + u(x,t)^3, \quad x \in \operatorname{int} \Gamma,\ t \in \mathbb{R},$$

with Kirchhoff boundary conditions at the vertex points and an odd integer k.



Existence of breathers:

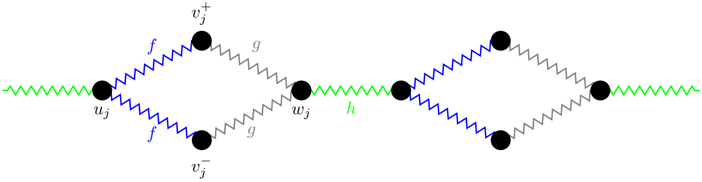
- real pulse solutions of amplitude $\mathcal{O}(\sqrt{\varepsilon})$, **symmetric** in the upper and lower semicircle,
- $u(x,t) = u(x, t + \frac{4\pi}{k})$, for all $t, x \in \mathbb{R}$,
- $u(t,x) \to 0$ as $|x| \to \infty$ exponentially fast for all $t \in \mathbb{R}$.

Discrete version of the necklace graph

Discrete Klein Gordon system

$$\begin{aligned}
\partial_t^2 u_j &= f(v_j^+ - u_j) + f(v_j^- - u_j) - h(u_j - w_{j-1}) - r_u(u_j),\\
\partial_t^2 v_j^+ &= g(w_j - v_j^+) - f(v_j^+ - u_j) - r_v(v_j^+),\\
\partial_t^2 v_j^- &= g(w_j - v_j^-) - f(v_j^- - u_j) - r_v(v_j^-),\\
\partial_t^2 w_j &= h(u_{j+1} - w_j) - g(w_j - v_j^+) - g(w_j - v_j^-) - r_w(w_j),
\end{aligned}$$

with relative displacement coordinates $u_j, v_j^\pm, w_j$ and interaction forces f, g, h, local forces r_u, r_v, r_w, for $j \in \mathbb{Z}$ on the discrete graph

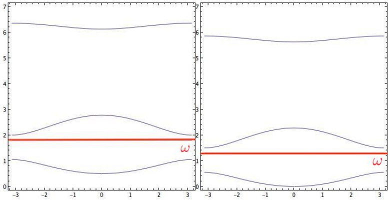


Nonlinearity: expansions $f(x) = f_1 x + f_2 x^2 + \ldots$

Discrete non-symmetric breathers

Spectral situation:

- Eigenvalue ω with anti-symmetric eigenstates
- Eigenstates that are not present in the initial data will not be excited at any time!



Left: $f_1 = 1$, $g_1 = 0.3$, $h_1 = 2$ (FPU);

Right: $f_1 = 1$, $g_1 = 0.3$, $h_1 = 2$ and $r_1 = 0.5$ (dKG)

Non-symmetric breathers bifurcate from a simple eigenvalue (Crandall-Rabinowitz theorem), if the non-resonance condition $m^2\omega^2 \notin \sigma_{ac}$ is satisfied for all $m \in \mathbb{N}_0$.

Future work: Transferring these ideas to the metric graph

Thank you.