On the notion of effective impedance via ordered fields

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Definition

Let (V, μ) be a locally finite weighted graph without isolated points (weights are positive/non-negative). Then for any function $f : V \to \mathbb{R}$, the function, defined by

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_y (f(y) - f(x))\mu_{xy},$$

is called the *(classical) Laplace operator* on (V, μ).

- Alexander Grigoryan. *Introduction to Analysis on Graphs*, 2018.

Fact

The Laplace operator on graphs is related with electric networks with resistors (since resistors have positive resistance) and direct current. The concept of *network* and *effective resistance of the network* is introduced in this case.

- David A. Levin, Yuval Peres, Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. 2009.
- Doyle P.G., Snell J.L. *Random walks and electric networks*. 1984.

Question

What about electric networks with other elements (coils, capacitors, and resistors)?
All together they are called *impedances*. Note, that current in this case should be alternating.

- R. P. Feynman. *The Feynman lectures on physics, Volume 2: Mainly Electromagnetism and Matter*. 1964
- O. Brune. *Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency*. Thesis (Sc. D.). Massachusetts Institute of Technology, Dept. of Electrical Engineering, 1931.

Definition

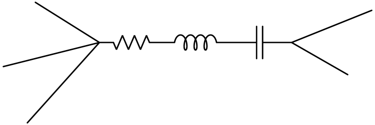
Network is a (finite) connected graph (V, E) where

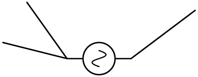
- to each edge $xy \in E$ the triple of non-negative numbers R_{xy} (resistance), L_{xy} (inductance), and C_{xy} (capacitance) is associated;
- two fixed vertices a_0, a_1 "are connected to a source of alternating current".

The *impedance* of the edge xy is

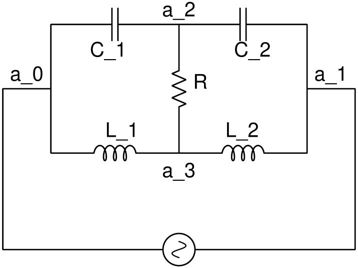
$$z_{xy} = R_{xy} + i\omega L_{xy} + \frac{1}{i\omega C_{xy}},$$

where ω is the frequency of the current.





Example


a_2
C_1
C_2
a_0
R
a_1
L_1
L_2
a_3

a_2
$\frac{1}{i\omega C_1}$
$\frac{1}{i\omega C_2}$
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R
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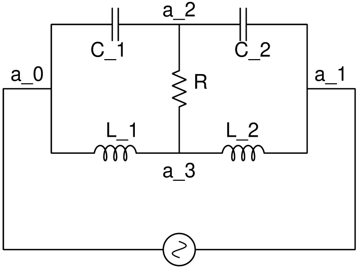
$i\omega L_1$
$i\omega L_2$
a_3


a_2
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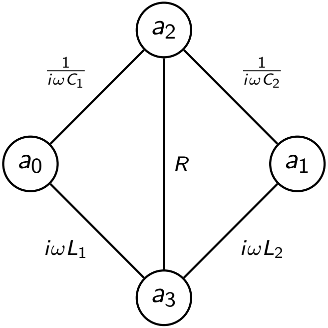
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By Ohm's complex law and Kirchhoff's complex law, the voltage $v(x)$ as a function on V satisfies the following *Dirichlet problem*:

$$\begin{cases} \Delta_\rho v(x) := \sum_{y:y\sim x}(v(x)-v(y))\rho_{xy} = 0 \text{ on } V\setminus\{a_0,a_1\}, \\ v(a_0)=0, v(a_1)=1, \end{cases} \quad (1)$$

where $\rho_{xy} = \frac{1}{z_{xy}}$ is the *admittance* of the edge xy. ρ_{xy} is hence a *complex-valued weight* of xy. Operator Δ_ρ is called the *Laplace operator*.

Note, that if $|V| = n$, then (1) is a $n \times n$ system of linear equations.

Definition

Effective impedance of the network is

$$Z_{eff} = \frac{1}{\sum_{x:x\sim a_0} v(x)\rho_{xa_0}} = \frac{1}{\mathcal{P}_{eff}},$$

where $v(x)$ is a solution of the Dirichlet problem. $\mathcal{P}_{eff}$ is called effective admittance and it is equal to the current through a_0 due to the unit voltage on the source.

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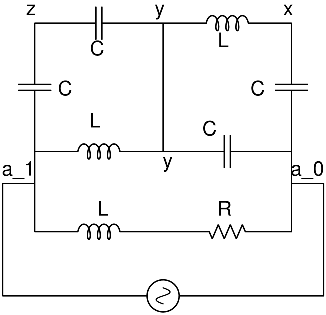
There are easy examples, when the Dirichlet problem has no solution or has multiple solutions



- In the case $\omega = \sqrt{\frac{2}{LC}}$ the Dirichlet problem has infinitely many solutions $(v(x), v(y), v(z)) = (-2\tau + 1, 2\tau - 1, \tau)$;
- In the case $\omega = \sqrt{\frac{1}{3LC}}$ the Dirichlet problem has no solutions.

Question

How to define $\mathcal{P}_{eff}$ in this cases?

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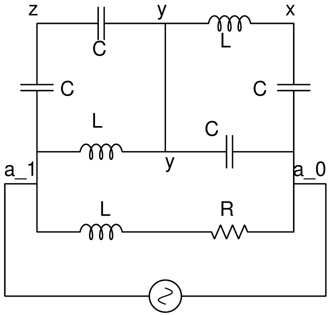

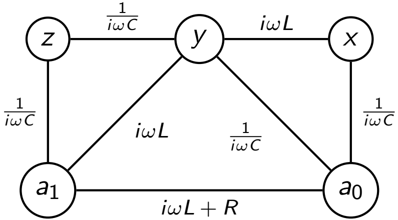

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Main problem

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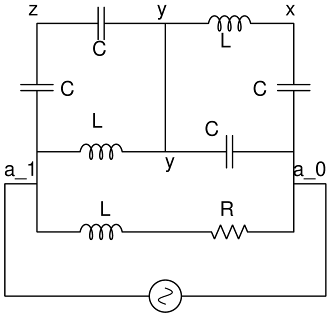

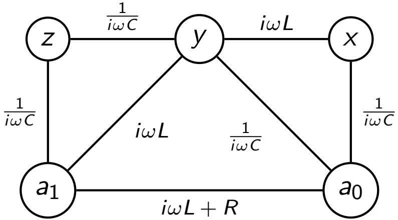

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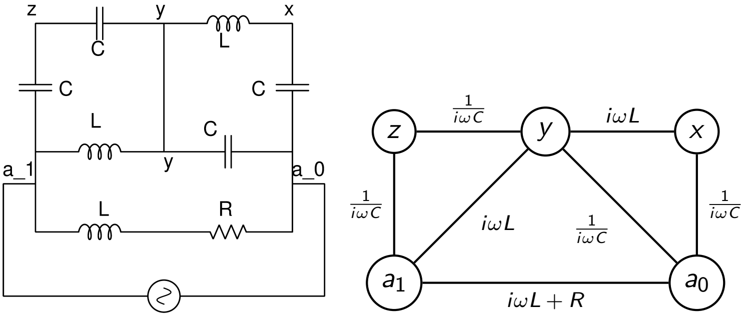

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Theorem 1

In the case of existence of multiple solutions of the Dirichlet problem (1) the value of $\mathcal{P}_{eff}$ does not depend on the choice of solution.

Proof.

If one writes the Dirichlet problem (1) in the form

$$\begin{cases} A\hat{v} = b, \\ v(a_0) = 0, v(a_1) = 1, \end{cases}$$

where A is a symmetric $(n-2) \times (n-2)$ matrix, then $\mathcal{P}_{eff} = (Ae - b)^T \hat{v} + \rho_{a_1 a_0}$. And for any two solutions $\hat{v}_1, \hat{v}_2$, we have:

$$\begin{aligned} (Ae - b)^T \hat{v}_1 =& e^T A^T \hat{v}_1 - b^T \hat{v}_1 = e^T A \hat{v}_1 - (A\hat{v}_2)^T \hat{v}_1 = e^T b - \hat{v}_2^T A^T \hat{v}_1 \\ =& e^T A \hat{v}_2 - \hat{v}_2^T A \hat{v}_1 = e^T A \hat{v}_2 - \hat{v}_2^T A \hat{v}_2 = e^T A^T \hat{v}_2 - \hat{v}_2^T A^T \hat{v}_2 \\ =& e^T A^T \hat{v}_2 - (A\hat{v}_2)^T \hat{v}_2 = e^T A^T \hat{v}_2 - b^T \hat{v}_2 = (Ae - b)^T \hat{v}_2. \end{aligned}$$

□

In the case of lack of solution we set by definition $\mathcal{P}_{eff} = +\infty$.

Conservation of the complex power

Note that for

$$z_{xy} = R_{xy} + i\omega L_{xy} + \frac{1}{i\omega C}$$

we have $\operatorname{Re} z_{xy} \geq 0$. Also, $\operatorname{Re} \rho_{xy} \geq 0$.
By physical meaning, $\operatorname{Re} Z_{eff}$ and $\operatorname{Re} \mathcal{P}_{eff}$ should be non-negative.

Theorem 2

$$\mathcal{P}_{eff} = \frac{1}{2} \sum_{x \sim y} |v(x) - v(y)|^2 \rho_{xy}.$$

Key point of the proof (Green's formula)

For any two functions f, g on V

$$\sum_{x \in V} \Delta_z f(x) \overline{g(x)} = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f) \overline{(\nabla_{xy} g)} \rho_{xy}, \qquad (2)$$

where $\overline{g(x)}$ is a complex conjugation of $g(x)$.

Corollary

$$\operatorname{Re} \mathcal{P}_{eff} \geq 0.$$

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$$\operatorname{Re} \mathcal{P}_{eff} \geq 0.$$

Note that all ρ_{xy} depend on ω. Therefore one can consider $\mathcal{P}_{eff}(\omega)$ as function on frequency.

Conjecture

$\mathcal{P}_{eff}(\omega)$ depends continuously on $\omega \geq 0$.

Difficulty is in the case when solution of the Dirichlet problem does not exist or is not unique.
Questions:

- is it possible that the solution of the Dirichlet problem (1) $v(\omega)$ has no limit, when $\omega \to \omega_0$, but the Dirichlet problem (1) for $\omega = \omega_0$ still has (multiple) solutions?
- is it possible that the solution of the Dirichlet problem (1) $v(\omega)$ has no limit, but $\mathcal{P}_{eff}$ has limit when $\omega \to \omega_0$?

In domain, where the solution of the Dirichlet problem is unique, $\mathcal{P}_{eff}(\omega)$ is a rational function of ω.

Alternative approach to definition of $\mathcal{P}_{eff}(\omega)$

We put $\lambda = i\omega$ and consider ρ_{xy} as a rational function of λ

$$\rho_{xy} = \frac{i\omega}{L_{xy}(i\omega)^2 + R_{xy}i\omega + \frac{1}{C_{xy}}} = \frac{\lambda}{L_{xy}\lambda^2 + R_{xy}\lambda + \frac{1}{C_{xy}}}$$

with real coefficients.

Let us denote by $\mathbb{R}(\lambda)$ the set of all rational functions on λ with real coefficients.

Definition

$\mathbb{R}(\lambda)$ is an ordered field, where the **total order** $\succ$ is defined as follows:

$$f(\lambda) = \frac{a_n\lambda^n + \cdots + a_1\lambda + a_0}{b_m\lambda^m + \cdots + b_1\lambda + b_0} \succ 0, \text{ if } \frac{a_n}{b_m} \succ 0$$

and

$$f(\lambda) \succ g(\lambda), \text{ if } f(\lambda) - g(\lambda) \succ 0.$$

We will write

$$f(\lambda) \succeq g(\lambda), \text{ if } f(\lambda) - g(\lambda) \succ 0 \text{ or } f(\lambda) = g(\lambda).$$

The ordered field $\mathbb{R}(\lambda)$ is non-Archimidean: $\lambda \succ n$ for any $n = \underbrace{1 + \cdots + 1}_{n}$.

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Consider now a graph (V, E) with edge weights $\rho \succ 0$, $\rho \in K$, where K is an ordered field (for example, $K = \mathbb{R}(\lambda)$).

Dirichlet problem

$$\begin{cases} \Delta_\rho v(x) := \sum_{y: y\sim x}(v(x) - v(y))\rho_{xy} = 0 \text{ on } V \setminus \{a_0, a_1\}, \\ v(a_0) = 0 \in K, v(a_1) = 1 \in K. \end{cases}$$

Theorem 3

The Dirichlet problem has always a unique solution v over the ordered field K and $0 \preceq v(x) \preceq 1$ for any $x \in V$. Consequently, the effective admittance

$$\mathcal{P}_{eff} = \sum_{x: x\sim a_0} v(x)\rho_{xa_0}$$

is always well-defined, and $\mathcal{P}_{eff} \succeq 0$.

In particular, in case $K = \mathbb{R}(\lambda)$ this gives unique effective admittance $\mathcal{P}_{eff}(\lambda)$ for any electrical network as rational function of $\lambda = i\omega$.

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A maximum/minimum principle

Let Ω be a non-empty subset of V. Then, for any function $v : V \to K$, that satisfies $\Delta_\rho v(x) = 0$ on $V \setminus \Omega$, we have

$$\max_{V\setminus\Omega} v \preceq \max_{\Omega} v \text{ and } \min_{V\setminus\Omega} v \succeq q \min_{\Omega} v.$$

A maximum/minimum principle holds for the Dirichlet problem over an ordered field and gives a uniqueness of the solution. This implies existence due to the general theory of finite dimensional linear operators.

Theorem 4

Let v be a solution (over a field K) of the Dirichlet problem for finite network, whose weights are positive elements from ordered field K.Then for any other function $f : V \to K$ such that $f(a_0) = 0, f(a_1) = 1$, the following inequality holds:

$$\sum_{xy \in E} (v(x) - v(y))^2 \rho_{xy} \preceq \sum_{xy \in E} (f(x) - f(y))^2 \rho_{xy}.$$

Moreover,

$$\mathcal{P}_{eff} = \frac{1}{2} \sum_{x \sim y} (v(x) - v(y))^2 \rho_{xy}.$$

Sketch of the proof

Write f as $f = g + v$ and use Green's formula for graphs over an ordered field:

$$\sum_{x \in V} \Delta_\rho f(x) g(x) = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} f)(\nabla_{xy} g) \rho_{xy},$$

which looks exactly like classical Green's formula for weighted graphs.

Statement

If the determinant of the Dirichlet problem (1) is not zero for some ω_0, then $\mathcal{P}_{eff} = \mathcal{P}_{eff}(i\omega_0)$. In the last expression the l.h.s. is calculated, using the (unique) solution of complex Dirichlet problem for the case $\omega = \omega_0$, and r.h.s. is calculated, using theory of ordered field.

Conjecture

$\mathcal{P}_{eff} = \mathcal{P}_{eff}(i\omega_0)$ for any positive ω_0.

Thank you!