Quantum Graphs on Radially Symmetric Antitrees

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(joint work with A. Kostenko)

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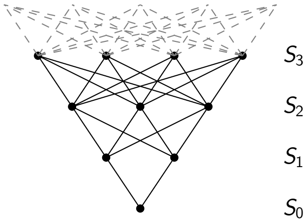
Definition

Let $(s_n)_n$ be a sequence with $s_0 = 1$ and $s_n \in \mathbb{N}$, $n \geq 1$. The **antitree** for $(s_n)_n$ is the (discrete) **graph** $\mathcal{A}_d = (\mathcal{V}, \mathcal{E})$ obtained as follows:

- For every $n \in \mathbb{N}$...
  1. Put s_n **new vertices**. Denote this vertex set by S_n.
  2. Then **connect** every vertex in S_n with every vertex in S_{n-1}.

<u>Ex.:</u> $s_n := n + 1$



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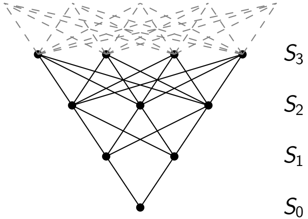

If every edge $e \in \mathcal{E}$ is assigned a **finite edge length** $0 < |e| < \infty$, then $\mathcal{A} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called a **metric antitree**.

$\Rightarrow$ Quantum Graphs **H** (= Laplacians) on metric antitrees

Motivation: QG's on different graph types?

- Finite graphs: (= finitely many edges)
  $\sigma(\mathbf{H})$ is **purely discrete** and the eigenvalues satisfy **Weyl's law**.

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Goal:

A model that can be fully analyzed - but still with "rich behavior"?

- **Random walks on antitrees** have diverse behavior!
  (Counter-examples to "Grigoryan's completeness theorem for graphs"; *Wojchiechowski 2011*)

The Kirchhoff Laplacian $\mathbf{H}$

Let $\mathcal{A}$ be a metric antitree and $L^2(\mathcal{A}) = \bigoplus_{e\in\mathcal{E}} L^2(0,|e|)$ its L^2-space. Then consider the **maximal operator** $\mathbf{H}_{\max} := \bigoplus_{e\in\mathcal{E}} \mathrm{H}_e$, where

$$\mathrm{H}_e = -\mathrm{d}^2/\mathrm{dx}_\mathrm{e}^2, \qquad \mathsf{dom}(\mathrm{H}_\mathrm{e}) = \mathrm{H}^2((0,|\mathrm{e}|)).$$

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Definition:

Define the **pre-minimal Laplacian** as $\mathbf{H}_0 := \mathbf{H}_{\max} \restriction \mathrm{dom}(\mathbf{H}_0)$ with domain

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Define the **Kirchhoff Laplacian H** by taking closure, $\mathbf{H} := \overline{\mathbf{H}_0}$.

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Additional assumption:

The antitree $\mathcal{A}$ is **radially symmetric**, i.e. for each $n \geq 0$, edges connecting the vertex sets S_n and S_{n+1} have the same length, say $\ell_n > 0$.

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Theorem (Kostenko–N.):

The **"symmetric part" $\mathbf{H}_{\mathrm{sym}}$** of **H** is equivalent to the Sturm-Liouville operator defined on $L^2([0, \mathcal{L}); \mu)$ by (here, $t_n := \sum_{j<n} \ell_j$, $\mathcal{L} = \sum_n \ell_n$)

$$\tau f := -\frac{1}{\mu(x)} \frac{d}{dx} \mu(x) \frac{d}{dx} f, \qquad \mu(x) = \sum_{n \geq 0} s_n s_{n+1} \mathbb{1}_{[t_n, t_{n+1})}(x),$$

and **Neumann BC** ($f'(0) = 0$) at $x = 0$.

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and **Neumann BC** ($f'(0) = 0$) at $x = 0$. Also, $\mathbf{H}$ decomposes as

$$\mathbf{H} = \mathbf{H}_{\mathrm{sym}} \oplus \bigoplus_{n\geq 1} \mathbf{h}_n,$$

where $\mathbf{h}_n$, $n \geq 1$ are equivalent to regular, s.a. Sturm-Liouville operators.

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Basic Questions:

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Moreover, if **H** is **not self-adjoint**, then $n_{\pm}(\mathbf{H}) = 1$.

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Moreover, if **H** is **not self-adjoint**, then $n_\pm(\mathbf{H}) = 1$.

The **symmetry assumption** is crucial. We can construct non-symmetric, **finite volume antitress** with $n_\pm(\mathbf{H}) = +\infty$!

The Finite Volume Case (= $\mathbf{H}$ is not self-adjoint)

Theorem (Kostenko–N.):

(i) Self-adjoint extensions form a one-parameter family $\mathbf{H}_\theta$, $\theta \in [0, \pi)$ given by **boundary conditions at "infinity"**

$$\cos(\theta) f(\infty) = \sin(\theta) f'(\infty), \qquad \theta \in [0, \pi), \tag{0.1}$$

where $f(\infty) := \lim_{|x| \to \mathcal{L}} f(x)$ and $f'(\infty) := \lim_{r \to \mathcal{L}} \sum_{|x|=r} f'(x)$.

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(ii) The spectrum of $\mathbf{H}_\theta$ is **purely discrete** for all θ and eigenvalues satisfy **Weyl's law**

$$N(\lambda; \mathbf{H}_\theta) = \frac{\mathrm{vol}(\mathcal{A})}{\pi} \sqrt{\lambda}(1 + o(1)), \quad \lambda \to +\infty.$$

Here, $N(\lambda; \mathbf{H}_\theta) = \#(\text{eigenvalues} \leq \lambda)$ is the **ev. counting function**.

The Infinite Volume Case (= **H** is s.a.): Basic properties

Theorem (Kostenko–N.):

(i) The spectrum of **H** is **purely discrete** if and only if

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(ii) $\mathbf{H}^{-1}$ is **trace class** (i.e., $\sum_{\lambda\in\sigma(\mathbf{H})} |\lambda|^{-1} < \infty$) if and only if

$$\sum_{n\ge 1} s_n s_{n+1} \ell_n^2 < \infty \quad \text{and} \quad \sum_{n\ge 0} \frac{\ell_n}{s_n s_{n+1}} \sum_{k=0}^{n-1} s_k s_{k+1} \ell_k < \infty.$$

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- Idea: Apply results from **spectral theory of Krein strings**!

Similarly: characterization of invertibility, spectral gap estimates,...

I. S. Kac & M. G. Krein, *Criteria for the discreteness of the spectrum of a singular string*, Izv. VUZov, Matematika, no. 2 (3), 136–153 (1958).

I. S. Kac & M. G. Krein, *On the spectral functions of the string*, AMS, 1974.

The Infinite Volume Case ($=$ $\mathbf{H}$ is s.a.): Spectral Types

$$\boxed{\sigma(\mathbf{H}) = \sigma_{\mathrm{ac}}(\mathbf{H}) \cup \sigma_{\mathrm{pp}}(\mathbf{H}) \cup \sigma_{\mathrm{sc}}(\mathbf{H})}$$

Recall the decomposition: $\mathbf{H} = \mathbf{H}_{\mathrm{sym}} \oplus \bigoplus_{n\geq 1} \mathbf{h}_n$

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- ... however, **"typically" no absolutely continuous spectrum**!

Theorem (Kostenko–N.):

Suppose the sets $\{\ell_n\}_{n\geq 0}$ and $\{\frac{s_{n+2}}{s_n}\}_{n\geq 0}$ are **finite** and $\liminf_{n\geq 0} \frac{s_{n+2}}{s_n} > 1$.

Then: $\sigma_{\mathrm{ac}}(\mathbf{H}) \neq \emptyset$ if and only if $\{(\ell_n, \frac{s_{n+2}}{s_n})\}_{n\geq 0}$ is eventually periodic.

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C. Remling, *The absolutely continuous spectrum of Jacobi matrices*, Ann. Math. **174**, no. 1, 125–171 (2011).

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Theorem (Kostenko–N.):

Assume that $\inf_n \ell_n > 0$. If

$$\sup_n \ell_n = \infty \qquad \text{and} \qquad \liminf_{n\geq 0} \frac{s_{n+2}}{s_n} > 1,$$

then $\sigma(\mathbf{H}) = \mathbb{R}_{\geq 0}$ and $\sigma_{ac}(\mathbf{H}) = \emptyset$.

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- Idea: Use **"Szegö's theorem"** for **Krein strings**

Bessonov&Denisov, *A spectral Szegő theorem on the real line*, preprint, arXiv:1711.05671 (2017).

Example: Polynomial antitrees

Let $q \in \mathbb{N}$ and $s > 0$. Consider the **polynomial antitree** $\mathcal{A}^{q,s}$ given by

$$s_n = (n+1)^q, \qquad \ell_n = (n+1)^{-s}.$$

Theorem (Kostenko–N.):

Let $\mathbf{H}$ be the Kirchhoff-Laplacian on $\mathcal{A}^{q,s}$. Then

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- If $s < 1$, then $\sigma_{\mathrm{ac}}(\mathbf{H}) = \mathbb{R}_{\geq 0}$.
- $\mathbf{H}$ is **invertible** if and only if $s \geq 1$.
- The spectrum $\sigma(\mathbf{H})$ is **purely discrete** if and only if $s > 1$.

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- $\mathbf{H}^{-1}$ is **trace class** if and only if $s > q + \frac{1}{2}$.

Thank you for your attention!

A. Kostenko and N. Nicolussi, *Quantum Graphs on Radially Symmetric Antitrees*, submitted, arXiv:1901.05404 (2019).

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