An eigenvalue bound for the Kirchhoff-Laplacian on planar quantum graphs

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The combinatorial setting

Let $G = (V, E)$ be a finite, simple and connected graph with vertex set $V = V(E)$ and edge set $E = E(G)$. 

We shall write $u \sim v$ if $u, v \in V$ are adjacent. Let $E_v = \{ e \in E | e \text{ is incident to } v \}$. 

Let $m : V \rightarrow (0, \infty)$ denote a positive weight function on the vertex set, let $m(U) := \sum_{u \in U} m(u)$ for $U \subset V$. 

Consider the space $l^2_m(V; \mathbb{C})$ of functions $f : V \rightarrow \mathbb{C}$ equipped with the norm $| | f | |^2_{l^2_m(V; \mathbb{C})} = \sum_{v \in V} m(v) | f(v) |^2$. 

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$$m(U) := \sum_{u \in U} m(u) \text{ for } U \subset V.$$

Consider the space $l^2_m(V; \mathbb{C}^d)$ of functions $f : V \to \mathbb{C}^d$ equipped with the norm

$$\|f\|_{l^2_m(V; \mathbb{C}^d)}^2 = \sum_{v \in V} m(u)|f(u)|^2.$$
The combinatorial setting

Let $\mu : E \to (0, \infty)$ be a positive edge weight. The (weighted) degree of a vertex $v \in V$ with respect to $\mu$ is

$$d^\mu_v = \sum_{e \in E_v} \mu(e), \quad d^\mu_{\text{max}} = \max_{v \in V} d^\mu_v \quad (1)$$
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On $l^2_m(V)$ we consider the (weighted) combinatorial Laplacian $\mathcal{L}$,

$$(\mathcal{L}f)(u) = \frac{1}{m(u)} \sum_{e = \{u, v\} \in E_v} \mu(e)(f(u) - f(v)), \quad u \in V$$
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and its associated quadratic form $q$ given by

$$q(f) = \sum_{e = \{u, v\} \in E} \mu(e)|f(u) - f(v)|^2.$$
The spectral gap of the combinatorial Laplacian

Lemma

The first positive eigenvalue of $\mathcal{L}$ is given by

$$\lambda_1(\mathcal{L}) = \inf_{\substack{f \in l^2_m(V) \setminus \{0\}, \|f\|_2^2, \|f\|_l^2_m(V)}} \frac{q(f)}{|f \perp_m 1_V|},$$

where

$$f \perp_m 1_V \iff \sum_{v \in V} m(v)f(v) = 0.$$
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The metric setting

Consider a metric graph

\[ \mathcal{G} = (G, l), \]

where

- \( G = (V, E) \) is a (combinatorial) connected, simple and finite graph
- \( l : E \to (0, \infty), e \mapsto l_e \) is a weight function on the edge set.
Consider a metric graph

\[ G = (G, l), \]

where
- \( G = (V, E) \) is a (combinatorial) connected, simple and finite graph and
- \( l : E \to (0, \infty), e \mapsto l_e \) is a weight function on the edge set.

We introduce the following quantities:
- the total length \( L = \sum_{e \in E} l_e \),
- the (weighted) degree \( d^l_v = \sum_{e \in E_v} l_e \) for \( v \in V \).
The metric setting

On the Hilbert space

\[ L^2(G; \mathbb{C}^d) := \bigoplus_{e \in E} L^2(0, l_e; \mathbb{C}^d), \quad \|f\|_{L^2(G; \mathbb{C}^d)}^2 = \sum_{e \in E} \|f_e\|_{L^2(0, l_e; \mathbb{C}^d)}^2. \]
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we consider the Kirchhoff-Laplacian \(-\Delta\) given by

\[ (-\Delta f)_e = -\frac{d^2}{dx_e^2} f_e, \quad \text{for } f \in \bigoplus_{e \in E} H^2(0, l_e), \]

such that

\[
\begin{cases}
  f \text{ is continuous in } v \\
  \sum_{v \in V} f_e'(v) = 0
\end{cases}
\]

for all \( v \in V. \)
Comparing the spectral gaps

Lemma

We choose the weights

\[ m(v) = d_v^l = \sum_{\ell \in E_v} l_{\ell}, \quad \mu(e) = \frac{1}{l_{\ell}} \]

then the first positive positive eigenvalues of the respective Laplacians satisfy the estimate

\[ \lambda_1(-\Delta) \leq 6 \lambda_1(\mathcal{L}). \]
Comparing the spectral gaps

**Lemma**

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\lambda_1(-\Delta) \leq 6 \lambda_1(\mathcal{L}).
\]

**Proof**

Consider edgewise affine functions and compare the Rayleigh quotients.
What is known in the manifold case?

**Theorem (Hassannezhad ’11)**

*Given a closed, oriented Riemannian manifold $M$ of genus $g > 0$ we have*

$$
\lambda_k(-\Delta_M) \leq C \frac{g + k}{\text{Vol}(M)}.
$$

*Previous results: Hersch ’70, Yang, Yau ’80, Korevaar ’93*
What is known in the (classical) combinatorial case?

\[(\mathcal{L}f)(u) = \sum_{v \sim u} (f(u) - f(v)), \; u \in V. \; (m \equiv 1, \mu \equiv 1)\]
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\[(L f)(u) = \sum_{v \sim u} (f(u) - f(v)), \ u \in V. \ (m \equiv 1, \mu \equiv 1)\]

Theorem (Spielman, Teng ’07)

\[\lambda_1(L) \leq C \frac{d_{\text{max}}}{|V|}, \ \text{if } G \text{ is planar.}\]
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\[\lambda_1(\mathcal{L}) \leq \text{poly}(d_{\text{max}}) \frac{g}{|V|}, \quad \text{if } G \text{ is of genus } g > 0\]
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**Theorem (Amini, Cohen-Steiner ’18)**

$$\lambda_k(L) \leq C \frac{d_{\text{max}}^2(g+k)}{|V|}, \text{ if } G \text{ is of genus } g \geq 0$$
Circle-Packings for Planar Graphs in the plane

The proof of Spielman and Teng used the following representation of planar graphs.

**Theorem (Koebe ’36, Andreev ’70, Thurston ’78)**

A graph $G = (V, E)$ is planar, iff there exists a family of closed disks $(D_v)_{v \in V}$ in the plane, such that the following holds for any two vertices $v \neq u$:

- If $v$ and $u$ are adjacent, the two disks $D_v$ and $D_u$ intersect at exactly one point.
- If $v$ and $u$ are not adjacent, the two disks $D_v$ and $D_u$ are disjoint.

In the above case $(D_v)_{v \in V}$ is called a circle-packing for $G$ in $\mathbb{R}^2$. 
Example of a circle-packing
Caps on the unit sphere $S^2$

We shall transfer the concept of circle packings to the unit sphere.

- A subset $k \subset S^2$ is called a circular line, if $k$ is a non-trivial intersection of the sphere $S^2$ and a hyperplane $H$ in $\mathbb{R}^3$.
Caps on the unit sphere $S^2$

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- A subset $k \subset S^2$ is called a circular line, if $k$ is a non-trivial intersection of the sphere $S^2$ and a hyperplane $H$ in $\mathbb{R}^3$.
- A connected, closed subset $C \subset S^2$ is called (spherical) cap, if its boundary is a circular line.
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- There is exactly one point $p(C) \in C$ of equal euclidian distance $r(C)$ to any point on the boundary of $C$. We call $p(C)$ the center and $r(C)$ the radius of $C$. 

The surface area of $C$ is $\pi \cdot r(C)^2$. 

**Fact**

The stereographic projection $\mathbb{R}^2 \rightarrow S^2$ maps disks in $\mathbb{R}^2$ to caps in $S^2$ and vice versa.
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A graph $G = (V, E)$ is planar, iff there exists a family of spherical caps $(C_v)_{v \in V}$, such that the following holds for any two vertices $v \neq u$:

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A first eigenvalue bound for the combinatorial Laplacian

Lemma

Assume $G$ is planar and there is a circle-packing $(C_v)_{v \in V}$ for $G$, such that

$$\sum_{v \in V} m(v) p(C_v) = 0,$$

then

$$\lambda_1(\mathcal{L}) \leq 8 \frac{d_{\text{max}}^\mu}{m(V)}$$

holds.
Proof of the Lemma

Idea

Use a circle-packing \((C_v)_{v \in V}\) representing the planar graph \(G\) to obtain a test function \(f\) in the vector-valued variational formulation.
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\[ q(f) = \sum_{e = \{u, v\} \in E} \mu(e) |p_u - p_v|^2 \]  

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\[ = 2 \sum_{v \in V} d_v^\mu r_v^2 \]

\[ \leq 2 d_{\text{max}}^\mu \sum_{v \in V} r_v^2. \]
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\[ = 2 \sum_{v \in V} d^\mu_v r_v^2 \]
\[ \leq 2 d^\mu_{\text{max}} \sum_{v \in V} r_v^2. \]
\[ \leq 8 d^\mu_{\text{max}} \]
Proof of the Lemma

Using these two estimates we obtain

\[
\frac{q(f)}{\|f\|_{L^2(V;\mathbb{C}^3)}} \leq 8 \frac{d_{\text{max}}}{m(V)}.
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By assumption

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By assumption

\[ \sum_{v \in V} m(v) f(v) = \sum_{v \in V} m(v) p_v = 0. \]

Therefore the min-max principle yields

\[ \lambda_1(\mathcal{L}) \leq 8 \frac{d_{\text{max}}}{m(V)}. \]
Uniqueness of circle packings

Question

How do we find a circle-packing, such that the orthogonality condition above is fulfilled?

Theorem (Koebe '36, Andreev '70, Thurston '78)

If \( G \) is maximal planar, then the circle packing representation in \( S^2 \) is unique up to conformal maps and reflections in \( S^2 \).

Canonical Approach

Find a conformal map \( S^2 \to S^2 \) that maps a given circle packing to a circle packing that satisfies the orthogonality condition.
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Canonical Approach

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Theorem (P. ’19)

Let $G$ be planar and $(C_v)_{v \in V}$ be a circle packing for $G$ in $S^2$. Assume that

$$m(V) > 2(m(u) + m(v))$$

for all $\{u, v\} \in E$. 


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$$m(V) > 2(m(u) + m(v)) \text{ for all } \{u, v\} \in E,$$

then there exists a conformal map $f : S^2 \to S^2$, such that

$$\sum_{v \in V} m(v)p(f(C_v)) = 0.$$
Idea of the proof

- After rescaling we may assume $m(V) = 1$. 

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- Consider a certain family of conformal maps $f_\alpha : S^2 \to S^2$ for $\alpha \in \mathbb{R}^3$, $|\alpha| < 1$, that moves the circles along the sphere in a convenient way.
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- Define the map $\Phi$ given by

$$\Phi(\alpha) = \sum_{v \in V} m(v) p(f_\alpha(C_v)).$$
Idea of the proof

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- Define the map $\Phi$ given by
  \[
  \Phi(\alpha) = \sum_{v \in V} m(v) p (f_\alpha(C_v)).
  \]
- Using a fixpoint argument one may show that $\Phi$ must be 0 for some $\alpha$ under the assumption on the vertex weight $m$. 
Corollary

Let $G$ be planar and assume that

$$m(V) > 2(m(v) + m(u)) \text{ for all } \{u, v\} \in E,$$

then we obtain the estimate

$$\lambda_1(L) \leq 8 \frac{d_{\text{max}}^\mu}{m(V)}.$$
Corollary

Let $G$ be planar and assume that

$$L > d_u^l + d_v^l \text{ for all } \{u, v\} \in E,$$

then we obtain the estimate

$$\lambda_1(-\Delta) \leq 24 \frac{d_{\text{max}}^l}{L},$$

where $\mu(e) = \frac{1}{l_e}$. 

An eigenvalue bound for planar quantum graphs
How to drop the condition $L > d_v^l + d_u^l$?

- Consider an arbitrary metric graph $\mathcal{G} = (G, l_G)$ over some combinatorial, connected and finite graph $G = (V(G), E(G))$, that is not necessarily simple.
How to drop the condition $L > d^l_v + d^l_u$?

- Consider an arbitrary metric graph $\mathcal{G} = (G, l_G)$ over some combinatorial, connected and finite graph $G = (V(G), E(G))$, that is not necessarily simple.

- Let $\mathcal{G}' = (G', l')$ be the subdivision graph obtained after dividing each edge into four edges of equal length. We shall write

$$V(G') = V_{\text{new}} \cup V_{\text{old}}.$$
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- Note

  $$L' = L, \quad \lambda_1(-\Delta_{\mathcal{G}'}) = \lambda_1(-\Delta_{\mathcal{G}}).$$
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Figure: Subdivision of $K_4$
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**Theorem (P.’19)**

*If* \( G = (G, l) \) *is a planar, finite, compact and connected metric graph, then we have the spectral bound*

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\lambda_1(-\Delta) \leq 192 \frac{d_{\text{max}}}{L}.
\]

**Remark**

The planarity assumption cannot be dropped! (Example: Complete equilateral graphs of constant length \( l \equiv 1 \).)

**To Do**

Generalize our results to higher order eigenvalues and graphs of higher genus.
Summary

Theorem (P.’19)

If $G = (G, l)$ is a planar, finite, compact and connected metric graph, then we have the spectral bound

$$\lambda_1(-\Delta) \leq 192 \frac{d^\mu}{L} \max.$$

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If $G = (G, l)$ is a planar, finite, compact and connected metric graph, then we have the spectral bound

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To Do

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Thank you for your attention!