An eigenvalue bound for the Kirchhoff-Laplacian on planar quantum graphs

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The combinatorial setting

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Consider the space $l^2_m(V; \mathbb{C}^d)$ of functions $f : V \to \mathbb{C}^d$ equipped with the norm

$$||f||^2_{l^2(V;\mathbb{C}^d)} = \sum_{v \in V} m(u)|f(u)|^2.$$

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Let $\mu : E \to (0, \infty)$ be a positive edge weight. The (weighted) degree of a vertex $v \in V$ with respect to μ is

$$d_v^\mu = \sum_{e \in E_v} \mu(e), \qquad d_{\max}^\mu = \max_{v \in V} d_v^\mu \tag{1}$$

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On $l_m^2(V)$ we consider the (weighted) combinatorial Laplacian $\mathcal{L}$,

$$(\mathcal{L}f)(u) = \frac{1}{m(u)} \sum_{e=\{u,v\} \in E_v} \mu(e)(f(u) - f(v)), \ u \in V$$

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and its associated quadratic form q given by

$$q(f) = \sum_{e=\{u,v\} \in E} \mu(e)|f(u) - f(v)|^2.$$

The spectral gap of the combinatorial Laplacian

Lemma

The first positive eigenvalue of $\mathcal{L}$ is given by

$$\lambda_1(\mathcal{L}) = \inf_{\substack{f \in l^2_m(V) \setminus \{0\}, \\ f \perp_m \mathbf{1}_V}} \frac{q(f)}{||f||^2_{l^2_m(V)}},$$

where

$$f \perp_m \mathbf{1}_V \Leftrightarrow \sum_{v \in V} m(v) f(v) = 0.$$

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The metric setting

Consider a metric graph

$$\mathcal{G} = (G, l),$$

where

- $G = (V, E)$ is a (combinatorial) connected, simple and finite graph and
- $l : E \to (0, \infty), e \mapsto l_e$ is a weight function on the edge set.

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We introduce the following quantities:

- the total length $L = \sum_{e \in E} l_e$,
- the (weighted) degree $d_v^l = \sum_{e \in E_v} l_e$ for $v \in V$.

The metric setting

On the hilbert space

$$L^2(\mathcal{G};\mathbb{C}^d) := \bigoplus_{e\in E} L^2(0,l_e;\mathbb{C}^d), \;\; ||f||^2_{L^2(\mathcal{G};\mathbb{C}^d)} = \sum_{e\in E} ||f_e||^2_{L^2(0,l_e;\mathbb{C}^d)}.$$

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we consider the Kirchhoff-Laplacian $-\Delta$ given by

$$(-\Delta f)_e = -\frac{\mathrm{d}^2}{\mathrm{d}x_e^2} f_e, \text{ for } f \in \bigoplus_{e\in E} H^2(0,l_e),$$

such that

$$\begin{Bmatrix} f \text{ is continuous in } v \\ \sum_{v\in V} f_e'(v) = 0 \end{Bmatrix} \text{ for all } v \in V.$$

Comparing the spectral gaps

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We choose the weights

$$m(v) = d_v^l = \sum_{e \in E_v} l_e, \qquad \mu(e) = \frac{1}{l_e}$$

then the first positive positive eigenvalues of the respective Laplacians satisfy the estimate

$$\lambda_1(-\Delta) \leq 6\,\lambda_1(\mathcal{L}).$$

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Proof

Consider edgewise affine functions and compare the Rayleigh quotients.

What is known in the manifold case?

Theorem (Hassannezhad '11)

Given a closed, oriented Riemannian manifold M of genus $g > 0$ we have

$$\lambda_k(-\Delta_M) \leq C\frac{g+k}{Vol(M)}.$$

Previous results: Hersch '70, Yang, Yau '80, Korevaar '93

What is known in the (classical) combinatorial case?

$$(\mathcal{L}f)(u) = \sum_{v \sim u} (f(u) - f(v)), \;\; u \in V. \;\; (m \equiv 1, \mu \equiv 1)$$

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Theorem (Amini, Cohen-Steiner '18)

$$\lambda_k(\mathcal{L}) \leq C \frac{d_{\max}^2 (g+k)}{|V|} \text{ if } G \text{ is of genus } g \geq 0$$

Circle-Packings for Planar Graphs in the plane

The proof of Spielman and Teng used the following representation of planar graphs.

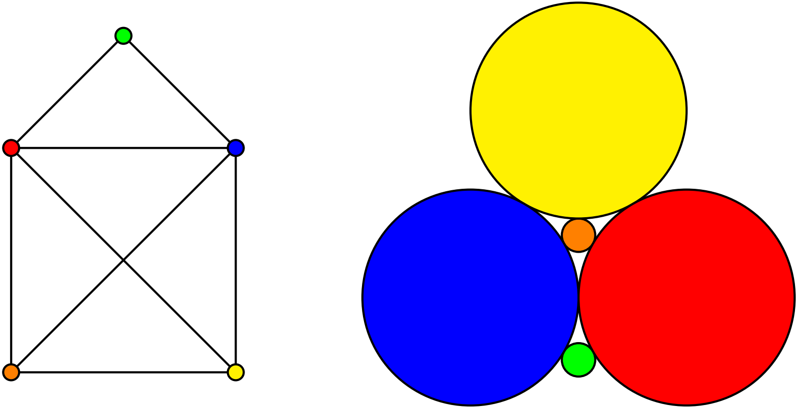
Theorem (Koebe '36, Andreev '70, Thurston '78)

A graph $G = (V, E)$ is planar, iff there exists a family of closed disks $(D_v)_{v \in V}$ in the plane, such that the following holds for any two vertices $v \neq u$:

- *If v and u are adjacent, the two disks D_v und D_u intersect at exactly one point.*
- *If v and u are not adjacent, the two disks D_v and D_u are disjoint.*

In the above case $(D_v)_{v \in V}$ is called a circle-packing for G in $\mathbb{R}^2$.

Example of a circle-packing



Caps on the unit sphere S^2

We shall transfer the concept of circle packings to the unit sphere.

- A subset $k \subset S^2$ is called a circular line, if k is a non-trivial intersection of the sphere S^2 and a hyperplane H in $\mathbb{R}^3$.

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Fact

The stereographic projection $\mathbb{R}^2 \to S^2$ maps disks in $\mathbb{R}^2$ to caps in S^2 and vice versa.

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A graph $G = (V, E)$ is planar, iff there exists a family of spherical caps $(C_v)_{v \in V}$, such that the following holds for any two vertices $v \neq u$:

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In the above case $(C_v)_{v \in V}$ is called a circle-packing for G in S^2.

A first eigenvalue bound for the combinatorial Laplacian

Lemma

Assume G is planar and there is a circle-packing $(C_v)_{v\in V}$ for G, such that

$$\sum_{v\in V} m(v)p(C_v) = 0,$$

then

$$\lambda_1(\mathcal{L}) \leq 8\frac{d^{\mu}_{\max}}{m(V)}$$

holds.

Proof of the Lemma

Idea

Use a circle-packing $(C_v)_{v \in V}$ representing the planar graph G to obtain a test function f in the vector-valued variational formulation.

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Using these two estimates we obtain

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Therefore the min-max principle yields

$$\lambda_1(\mathcal{L}) \leq 8\frac{d^{\mu}_{\max}}{m(V)}.$$

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Question

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If G is maximal planar, then the circle packing representation in S^2 is unique up to conformal maps and reflections in S^2.

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Canonical Approach

Find a conformal map $S^2 \to S^2$ that maps a given circle packing to a circle packing that satisfies the orthogonality condition.

A Uniformization Theorem

Theorem (P. '19)

Let G be planar and $(C_v)_{v\in V}$ be a circle packing for G in S^2. Assume that

$$m(V) > 2(m(u)+m(v)) \text{ for all } \{u,v\} \in E,$$

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then there exists a conformal map $f : S^2 \to S^2$, such that

$$\sum_{v\in V} m(v)p\big(f(C_v)\big) = 0.$$

Idea of the proof

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$$\Phi(\alpha) = \sum_{v \in V} m(v) p\left(f_\alpha(C_v)\right).$$

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$$\Phi(\alpha) = \sum_{v \in V} m(v) p\left(f_\alpha(C_v)\right).$$

- Using a fixpoint argument one may show that Φ must be 0 for some α under the assumption on the vertex weight m.

An eigenvalue bound for planar combinatorial graphs

Corollary

Let G be planar and assume that

$$m(V) > 2(m(v) + m(u)) \text{ for all } \{u, v\} \in E,$$

then we obtain the estimate

$$\lambda_1(\mathcal{L}) \leq 8\frac{d^{\mu}_{\max}}{m(V)}.$$

An eigenvalue bound for planar quantum graphs

Corollary

Let $\mathcal{G}$ be planar and assume that

$$L > d_u^l + d_v^l \text{ for all } \{u, v\} \in E,$$

then we obtain the estimate

$$\lambda_1(-\Delta) \leq 24 \frac{d_{\max}^{\mu}}{L},$$

where $\mu(e) = \frac{1}{l_e}$.

How to drop the condition $L > d_v^l + d_u^l$?

- Consider an arbitrary metric graph $\mathcal{G} = (G, l_G)$ over some combinatorial, connected and finite graph $G = (V(G), E(G))$, that is not necessarily simple.

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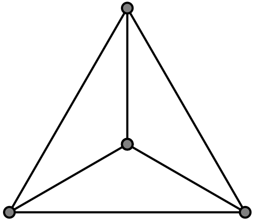

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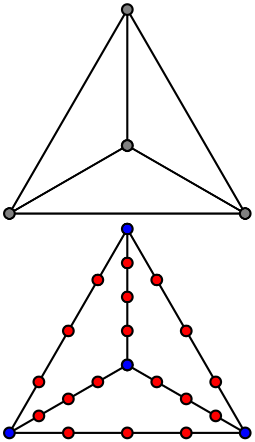

Figure: Subdivsion of K_4

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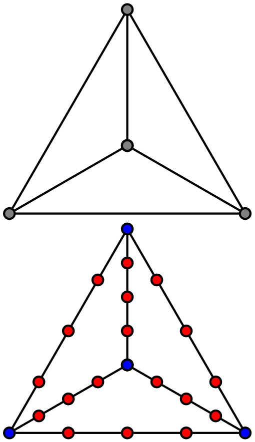

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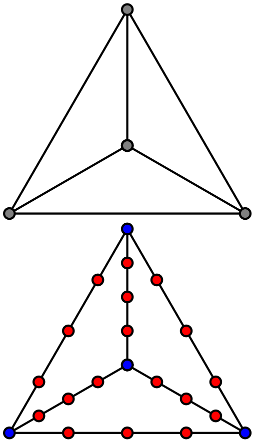
- Note

$$L' = L, \quad \lambda_1(-\Delta_{\mathcal{G}'}) = \lambda_1(-\Delta_{\mathcal{G}}).$$



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We have

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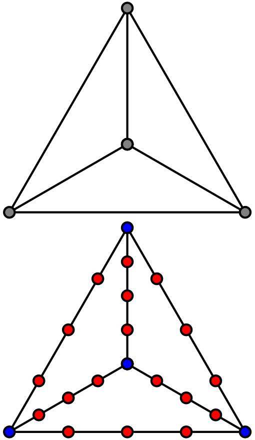

Figure: Subdivsion of K_4

How to drop the condition $L > d_v^l + d_u^l$?

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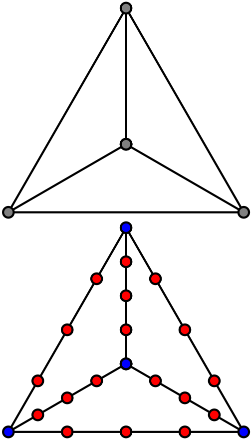
.



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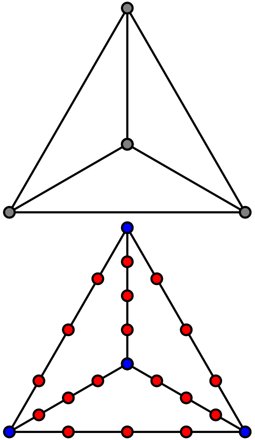

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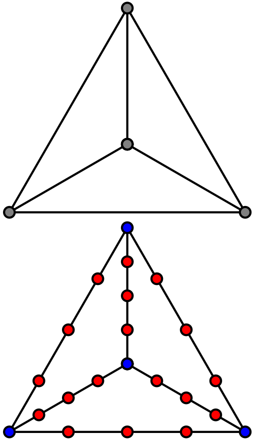

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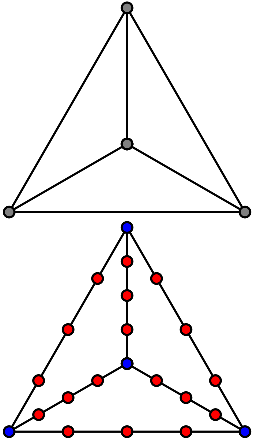
$$\lambda_1(-\Delta) \leq 24 \frac{d_{\max}^{\mu'}}{L} \qquad .$$



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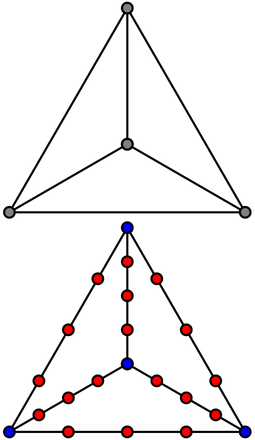

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Theorem (P.'19)

If $\mathcal{G} = (G, l)$ is a planar, finite, compact and connected metric graph, then we have the spectral bound

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To Do

Generalize our results to higher order eigenvalues and graphs of higher genus.

Thank you for your attention!