Weyl asymptotics of resonances and resonance states completeness for quantum graphs

Igor Popov

ITMO University
St. Petersburg, Russia
joint work with I.Blinova

Resonances. Introduction

An interesting example is given by the Helmholtz resonator
P.D.Hislop, A.Martinez, Scattering resonances of Helmholtz resonator, Indiana Univ. Math. J. 40 (1991), 767–788.
Consider the Laplace operator in bounded domain $\Omega^{in} \in \mathbb{R}^3$ with the Neumann or Dirichlet boundary condition. It is a self-adjoint operator with purely discrete spectrum. The set of its eigenfunctions is complete in $L_2(\Omega^{in})$. Consider now a perturbed problem, in which there is a small coupling window in the boundary connecting Ω^{in} and $\Omega^{ex} = \overline{\mathbb{R}^3 \setminus \Omega^{in}}$. This perturbation destroys the point spectrum of the initial operator. Eigenvalues move to the complex plane and become resonances (quasi-eigenvalues). The corresponding resonance states (formed from the eigenstates) satisfy the proper Helmholtz equation and the boundary condition but do not belong to $L_2(\mathbb{R}^3)$ (due to this reason they are not eigenfunctions, the integral of its square diverges at infinity). But what about completeness? The resonance states belong to $L_2(\Omega)$ for any bounded Ω. Is there such domain Ω that the resonance states are complete in $L_2(\Omega)$? We are interested in maximal domain. I believe that such domain is the convex hull of the scatterer (Ω^{in} with the boundary window). But at present, this conjecture has not yet been proved.
The simplest example of such scattering system is given by quantum graphs.

Resonances. Introduction

Several approaches:

Lax-Phillips scattering theory. P.D.Lax and R.S.Phillips. *Scattering theory*. Academic Press, New York (1967)

Complex scaling J.Sjöstrand, M.Zworski. Complex scaling and the distribution of scattering poles. J.Amer. Math. Soc. **4** (1991) 729-769.

Perturbation theory. S.Agmon. A perturbation theory of resonances. Comm. Pure Appl. Math. **51** (1998) 1255-1309. M.Rouleux. Resonances for a semi-classical Schrödinger operator near a non trapping energy level. Publ. RIMS, Kyoto Univ. **34** (1998) 487-523.

Functional model. S. V.Khrushchev, N.K.Nikol'skii, B.S.Pavlov. Unconditional bases of exponentials and of reproducing kernels, Complex Analysis and Spectral Theory (Leningrad, 1979/1980), Lecture Notes in Math., vol. 864, Springer-Verlag, Berlin␣New York, 214␣-335 (1981).

Asymptotic method. R.R.Gadyl'shin, Existence and asymptotics of poles with small imaginary part for the Helmholtz resonator, Russian Mathematical Surveys, 52 (1) (1997), 1–72.

Resonances. Introduction

In the case of a non-homogeneous string and in the related case of a 1-D Schrödinger equation, the completeness and the Riesz basis properties were a subject of intensive studies in connection with the Regge problem. There are several approaches to the formulation of the completeness problem.

One considers the family of root vectors of an operator that has resonances as its eigenvalues. This operator is essentially the generator of the semigroup for the associated wave equation:

M.G. Krein, A.A. Nudelman, On direct and inverse problems for frequencies of boundary dissipation of inhomogeneous string, Dokl. Akad. Nauk SSSR, 247 (5) (1979), 1046–1049.

M.G. Krein, A.A. Nudelman, Some spectral properties of a nonhomogeneous string with a dissipative boundary condition, J. Operator Theory 22 (1989), 369–395 (Russian).

The studies in this direction were continued by

S. Cox, E. Zuazua, The rate at which energy decays in a string damped at one end, Indiana Univ. Math. J., 44 (2) (1995), 545–573.

Resonances. Introduction

Another approach considers the completeness of resonant modes in suitable L_2-spaces on the intervals of the real line

M.A. Shubov, Asymptotics of resonances and eigenvalues for nonhomogeneous damped string. Asymptotic Analysis, 13 (1) 1996, 31–78.

S.V. Hrushchev, The Regge problem for strings, unconditionally convergent eigenfunction expansions, and unconditional bases of exponentials in $L_2(-T,T)$. Journal of Operator Theory, 14 (1) (1985), 67–85.

Graph model. Introduction

We consider quantum graphs of the following structure. Let us start with a finite compact metric graph Γ_0 and choose some subset of vertices of Γ_0, to be called external vertices, and attach one or more copies of $[0;\infty)$, to be called leads, to each external vertex; the point 0 in a lead is thus identified with the relevant external vertex. The thus extended graph Γ is the subject of the investigation. we will call the edges of Γ_0 as "edges"(or internal edges, E^{int} and other (infinite) edges of Γ as the "leads"(E^{ext}). Correspondingly, let V be the set of all vertices of Γ, let V^{ext} be the set of all external vertices, and let $V^{int} = V \setminus V^{ext}$; the elements of V^{int} will be called internal vertices.

Definition 0.1 *A vertex is named "external"if it has semi-infinite lead attached and "internal"in the opposite case.*

Definition 0.2 *An external vertex is named "balanced"if for this vertex the numbers of attached leads equals to the number of attached edges. If it is not balanced, we call it unbalanced.*

Graph model. Introduction

We consider one-dimensional free Schrödinger operator on each edge and lead (i.e. the second derivative: $H = -\frac{d^2}{dx^2}$). The domain of H consists of continuous functions on Γ, belonging to W_2^2 on each lead and edge satisfying the boundary conditions at boundary vertices (we assume the Dirichlet condition), coupling conditions at other vertices (we assume the Kirchhoff condition):

$$\sum_{e, v\in e} (-1)^{\kappa(e(v))} \frac{\partial \psi}{\partial x} = 0, \tag{1}$$

where $\kappa(e(v)) = 0$ for outgoing edge/lead e and $\kappa(e(v)) = 1$ for incoming edge e (we assumed earlier that all leads are outgoing).

Graph. Introduction

The standard definition of resonance is as follows

Definition 0.3 *We will say that $k \in \mathbb{C}, k \neq 0$, is a resonance of H (or, by a slight abuse of terminology, a resonance of Γ) if there exists a resonance eigenfunction (resonance state) $f, f \in L^2_{loc}\Gamma$, which satisfies the equation*

$$-\frac{d^2 f}{dx^2}(x) = k^2 f(x), \quad x \in \Gamma,$$

on each edge and lead of Γ, is continuous on Γ, satisfies the boundary conditions at boundary vertices, coupling conditions at other vertices, and the radiation condition $f(x) = f(0)e^{ikx}$ on each lead.

We denote the set of resonances as Λ. Below we will give one more, equivalent, definition of resonances in the framework of the Lax-Phillips scattering theory.

Graph. Asymptotics of resonances

We define the resonance counting function by

$$N(R) = \sharp\{k : k \in \Lambda, |k| \leq R\}, R > 0,$$

with the convention that each resonance is counted with its algebraic multiplicity taken into account. Note that the set R of resonances is invariant under the symmetry $k \to -\overline{k}$, so this method of counting yields, roughly speaking, twice as many resonances as one would obtain if one imposed an additional condition $\Re(k) \geq 0$. In particular, in the absence of leads, $N(R)$ equals twice the number of eigenvalues $\lambda \neq 0$ of H (counting multiplicities) with $\lambda \leq R^2$.

Graph. Asymptotics of resonances

There are works concerning to asymptotic of resonances in the complex plane.
E.B.Davies, A.Pushnitski. Non-Weyl resonance asymptotics for quantum graphs. Analysis and PDE 4 (2011), 729Ц756.
E.B. Davies, P. Exner, J. Lipovsky: Non-Weyl asymptotics for quantum graphs with general coupling conditions, J. Phys. A: Math. Theor. 43 (2010), 474013.
P. Exner, J. Lipovsky. Non-Weyl resonance asymptotics for quantum graphs in a magnetic field. Phys. Lett. A375 (2011), 805-807.

Graph. Asymptotics of resonances

If there are no leads then H has pure point spectrum, there are no resonances, but we can say that resonances are identified with eigenvalues of H, and it is known that for these eigenvalues, one has Weyl's law:

$$N(R) = \frac{2}{\pi}\mathrm{vol}(\Gamma_0)R + o(R), \mathrm{as} R \to \infty, \tag{2}$$

where $\mathrm{vol}(\Gamma_0)$is the sum of the lengths of the edges of Γ_0. We say that Γ (i.e. the corresponding graph with leads) is a Weyl graph, if the asymptotics (2) takes place for resonances of Γ.

The following theorem was proved by Davies, Pushnitski (2011).

Theorem 0.4 *One has*

$$N(R) = \frac{2}{\pi}WR + O(1), \mathrm{as} R \to \infty, \tag{3}$$

where the coefficient W satisfies $0 \leq W \leq \mathrm{vol}(\Gamma_0)$. One has $W = \mathrm{vol}(\Gamma_0)$ if and only if every external vertex of Γ is unbalanced.

Resonances and Lax-Phillips approach

Consider the Cauchy problem for the time-dependent Schrödinger equation on the graph Γ:

$$\begin{cases} i\hbar u'_t = Hu, \\ u(x,0) = u^0(x),\ x \in \Gamma. \end{cases} \tag{4}$$

The standard Lax-Phillips approach is applied to the wave (acoustic) equation. There is a close relation between the Schrödinger and wave cases.We will describe it briefly following (Lax and Phillips, 1971). Consider the Cauchy problem for the wave equation

$$\begin{cases} u''_{tt} = u''_{xx}, \\ u(x,0) = u_0(x), u'_t(x,0) = u_1(x), x \in \Gamma. \end{cases} \tag{5}$$

Let $\mathcal{E}$ be the Hilbert space of two-component functions (u_0, u_1) on the graph with finite energy

$$\|(u_0, u_1)\|^2_{\mathcal{E}} = 2^{-1} \int_\Gamma (|u'_0|^2 + |u_1|^2) dx.$$

The pair (u_0, u_1) is called the Cauchy data. Solving operator for problem (5), $U(t)$, $U(t)(u_0, u_1) = (u(x,t), u'_t(x,t))$, is unitary in $\mathcal{E}$.

Resonances and Lax-Phillips approach

Unitary group $U(t)|_{t\in\mathbb{R}}$ has two orthogonal (in $\mathcal{E}$) subspaces, D_- and D_+, called, correspondingly, incoming and outgoing subspaces.

Definition 0.5 *The outgoing (incoming) subspace $D_+(D_-)$ is a subspace of $\mathcal{E}$ having the following properties:*
(a) $U(t)D_+ \subset D_+$ *for* $t > 0$; $U(t)D_- \subset D_-$ *for* $t < 0$,
(b) $\cap_{t>0}U(t)D_+ = \{0\}$; $\cap_{t<0}U(t)D_- = \{0\}$
(c) $\overline{\cup_{t<0}U(t)D_+} = \mathcal{E}$, $\overline{\cup_{t>0}U(t)D_-} = \mathcal{E}$.

The incoming and outgoing subspaces are not unique. In our case we use the following choice. For the graph Γ, the subspace D_+ contains functions vanishing at Γ_0 (e.g. on all edges of finite length) and satisfying the radiation condition on all leads.
Let $P_\pm$ be the orthogonal projection of $\mathcal{E}$ onto the orthogonal complement of $D_\pm$.
Consider the semigroup $\{Z(t)\}|_{t\geq 0}$ of operators on $\mathcal{E}$ defined by

$$Z(t) = P_+U(t)P_-, \ \ t \geq 0.$$

Lax and Phillips proved the following theorem [**?**].

Theorem 0.6 *The operators* $\{Z(t)\}|_{t\geq 0}$ *annihilate* D_+ *and* D_-, *map the orthogonal complement subspace* $K = \mathcal{E} \ominus (D_- \oplus D_+)$ *into itself and form a strongly continuous semigroup* (*i.e.*, $Z(t_1)Z(t_2) = Z(t_1 + t_2)$ *for* $t_1, t_2 \geq 0$) *of contraction operators on* K. *Furthermore, we have* $s\text{-}\lim_{t\to\infty} Z(t) = 0$. *The space* $\mathcal{E}$ *can be represented isometrically as the Hilbert space of functions* $L_2(\mathbb{R}, N)$ *for some auxiliary Hilbert space* N *in such a way that* $U(t)$ *goes to translation to the right by* t *units and* D_+ *is mapped onto* $L_2(\mathbb{R}_+, N)$. *This representation is unique up to an isomorphism of* N.

Such a representation is called an outgoing translation representation. Analogously, one can obtain an incoming translation representation, i.e., if D_- is an incoming subspace with respect to the group $\{U(t)\}_{t\in\mathbb{R}}$ then there is a representation in which $\mathcal{E}$ is mapped isometrically onto $L_2(\mathbb{R}, N)$, $U(t)$ goes to translation to the right by t units and D_- is mapped onto $L_2(\mathbb{R}_-, N)$.

Resonances and Lax-Phillips approach

The Lax-Phillips scattering operator $\tilde{S}$ is defined as follows (it was proved that this definition is equivalent to the standard one). Suppose $W_+ : \mathcal{E} \to L_2(\mathbb{R}, N)$ and $W_- : \mathcal{E} \to L_2(\mathbb{R}, N)$ are the mappings of $\mathcal{E}$ onto the outgoing and incoming translation representations, respectively. The map $\tilde{S} : L_2(\mathbb{R}, N) \to L_2(\mathbb{R}, N)$ is defined by the formula

$$\tilde{S} = W_+(W_-)^{-1}.$$

For most purposes it is more convenient to work with the Fourier transforms F of the incoming and the outgoing translation representations, respectively, called the incoming spectral representation and the outgoing spectral representation. In the incoming (outgoing) spectral representation, $D_{-(+)}$ is represented by $H^2_{+(-)}(\mathbb{R}, N)$, i.e., by the space of boundary values on $\mathbb{R}$ of functions in the Hardy space $H^2(\mathbb{C}^{+(-)}, N)$ of vector-valued functions (with values in N) defined in the upper (lower) half-plane $\mathbb{C}^{+(-)}$.

$$S = F\tilde{S}F^{-1}.$$

The operator S is realized as the operator of multiplication by the operator-valued function $S(\cdot) : \mathbb{R} \to B(N)$, where $B(N)$ is the space of all bounded linear operators on N. $S(\cdot)$ is called the Lax-Phillips S-matrix.

Theorem 0.7 *(Lax and Phillips) (a) $S(\cdot)$ is the boundary value on $\mathbb{R}$ of an operator-valued function $S(\cdot) : \mathbb{C}^+ \to B(N)$ analytic in $\mathbb{C}^+$,*
(b) $\|S(z)\| \leq 1$ for every $z \in \mathbb{C}^+$,
(c) $S(E)$, $E \in \mathbb{R}$, is, pointwise, a unitary operator on N.

A conventional procedure allows one to construct the analytic continuation of $S(\cdot)$ from the upper half-plane to the lower half-plane:

$$S(z) = (S^*(\overline{z}))^{-1}, \ \ \Im z < 0.$$

Thus, $S(\cdot)$ is a meromorphic operator-valued function on the whole complex plane. Let B be the generator of the semigroup $Z(t)$: $Z(t) = \exp i\mathrm{B}t, t > 0$. In our case, the operator B acts as the second derivative on the edges of Γ. The domain of the operator consists of functions from the Sobolev space $W_2^2(\Gamma_0)$ satisfying the coupling conditions at the graph vertices and functions satisfying the radiation condition at leads. The the square roots of eigenvalues of B are called resonances and the corresponding root vectors are the resonance states. Note that this definition of resonance is in accordance with Definition 0.3.

Resonances and Lax-Phillips approach

There is a relation between the eigenvalues of B and the poles of the S-matrix.

Theorem 0.8 *If $\Im k < 0$, then k belongs to the point spectrum of* B *if and only if $S^*(\bar{k})$ has a non-trivial null space.*

Remark 0.9 *The theorem shows that a pole of the Lax-Phillips S-matrix at a point k in the lower half-plane is associated with an eigenvalue k^2 of the generator* B *of the Lax-Phillips semigroup $Z(t)$. In other words, resonance poles of the Lax-Phillips S-matrix correspond to eigenvalues of the Lax-Phillips semigroup with well defined eigenvectors belonging to the subspace $K = \mathcal{E} \ominus (D_- \oplus D_+)$, which is called the resonance subspace.*

Theorem 0.10 *There is a pair of isometric maps* $T_\pm : \mathcal{E} \to L_2(\mathbb{R}, N)$ (*the outgoing and incoming spectral representations*) *having the following properties:*

$$T_\pm U(t) = e^{ikt} T_\pm, \quad T_\pm D_\pm = H^2_\pm(N), \quad T_- D_+ = S H^2_+(N),$$

where $H^2_\pm(N)$ *is the Hardy space of the upper (lower) half-plane, the matrix-function* S *is an inner function in* $\mathbb{C}_+$, *and*

$$K_- = T_- K = H^2_+ \ominus S H^2_+, \quad T_- Z(t)|_K = P_{K_-} e^{ikt} T_-|_{K_-}.$$

Resonances and Lax-Phillips approach

The relation between the acoustic and the Schrödinger scattering matrix is given in (Lax, Phillips). Namely, it is necessary to consider the operator A^2:

$$A^2 = \begin{pmatrix} -\frac{1}{\hbar}H & 0 \\ 0 & -\frac{1}{\hbar}H \end{pmatrix}.$$

The operator A,

$$A = \begin{pmatrix} 0 & I \\ -\frac{1}{\hbar}H & 0 \end{pmatrix},$$

is the generator of $U(t)$ for the acoustic problem:

$$\frac{d}{dt}\left(U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right) = AU(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

One can see that A^2 acts as the Hamiltonian $-\frac{1}{\hbar}H$ on each component of the data of the acoustic problem. The relation between the Schrödinger (S^{Schr}) and the acoustic (S) scattering matrices:

$$S^{Schr}(z) = S(\sqrt{z}).$$

Functional model

The completeness of the system of the root vectors $e_n(\nu)$ in $H^2_+(N) \ominus SH^2_+(N)$ is related to the properties of the analytic operator-function S (characteristic function), more precisely, to its factorization.

Sz.-Nagy, B., Foias, C., Bercovici, H., Kerchy, L.: Harmonic Analysis of Operators on Hilbert Space, 2nd edition. Springer, Berlin (2010)

Nikol'skii, N.: Treatise on the shift operator: spectral function theory. Springer Science & Business Media, Berlin (2012).

Khrushchev, S.V., Nikol'skii, N.K., Pavlov, B.S.: Unconditional bases of exponentials and of reproducing kernels, Complex Analysis and Spectral Theory (Leningrad, 1979/1980). Lecture Notes in Math. 864, 214–335 (1981)

Definition 4.1. Operator Z acting in the space X is called complete, if the family of its root vectors is complete, i.e. if

$$\vee(\mathrm{Ker}(Z-\lambda I)^n = 0, \quad \lambda \in \sigma_p(Z), n \geq 1) = X.$$

Functional model

Definition 4.2. Blaschke-Potapov product (Potapov) is the following operator:

$$B(z) = \prod_n (b_{\lambda_n}(z) P_n + (I - P_n)),$$

where $\lambda_n, \lambda_n \in \sigma_p(C)$ is some ordering (with multiplicities) of the point spectrum of $Z(t)$, P_n is the corresponding orthogonal projectors in E, $b_{\lambda_n}(z)$ is the Blaschke factor:

$$b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}.$$

Each inner function S can be represented as $S = B\Theta$ where B is the Blaschke product and Θ is singular inner function, i.e. inner function having no zeros in the unit circle (or upper half-plane). As for scalar functions in the unit circle, there is simple criterion of absence of the singular inner factor (for operator case there is no such criterion):

$$\lim_{r \uparrow 1} \int_{|\zeta|=1} \log |\varphi(r\zeta)| dm(\zeta) = 0. \tag{6}$$

Functional model

Theorem 4.3. (Nikolskii) (Completeness criterion). Let S be an inner function, $H^2_+(N) \ominus SH^2_+(N)$, $Z = P_K U|_K$. The following statements are equivalent:
1. Operator Z is complete;
2. Operator Z^* is complete;
3. S is a Blaschke product.

Functional model

The problem simplifies considerably for operators of C_0 class (operator-functions having scalar multiples).

Definition 4.4. C_0 is a class of all non-unitary contractions Z (or dissipative operator for a half-plane) such that for each operator of the class, there exists function $\varphi, \varphi \neq 0$, annihilating the operator: $\varphi(Z) = 0$.

Lemma 4.5. Let $Z \in C_0$. There exists inner function m_Z such that $m_Z(Z) = 0$ and

$$\varphi \in H^\infty, \varphi(Z) = 0 \Rightarrow \varphi/m_Z \in H^\infty.$$

Remark 4.6. This function m_Z is a minimal annihilator.

Functional model

Theorem 4.7. Let $\varphi \in H^\infty$. The following statements are equivalent:
1. $\varphi(Z) = 0$.
2. There exists bounded analytic function ω, $\omega(\zeta) \in L(N), |\zeta| < 1,$ (or $\Im\zeta < 0$ for a half-plane) such that

$$S\omega = \omega S = \varphi I.$$

Remark 4.8. If $\mathrm{dim} N < \infty$ then the determinant d, $d(\zeta) = \mathrm{det} S(\zeta)$, $|\zeta| < 1$, is an annihilator for Z.

Functional model

For operators of C_0 class the completeness criterion becomes simpler.
Theorem 4.9. (Nikolskii) (completeness criterion 1). Let $Z \in C_0$. The following statements are equivalent:
1. Operator Z is complete;
2. Operator Z^* is complete;
3. m_Z is a Blaschke product.
One can see that for such operator, the completeness problem reduces to factorization of scalar function. Correspondingly, it is possible to use the criterion (6). As for the general case of the factorization problem, there is no effective criterion of singular factor absence.

Functional model

There is a simple criterion for the absence of the singular inner factor in the case $\dim N < \infty$ (in the general operator case there is no such simple criterion).

Theorem 0.11 *(Nikolskii). Let $\dim N < \infty$. The following statements are equivalent:*
1. S is a Blaschke-Potapov product;
2.

$$\lim_{r\to 1} \int_{C_r} \ln|\det S(k)| \frac{2i}{(k+i)^2} dk = 0, \tag{7}$$

where C_r is the image of $|\zeta| = r$ under the inverse Cayley transform.

The integration curve can be parameterized as $C_r = \{\mathsf{R}(r)e^{it} + i\mathsf{C}(r) \mid t \in [0, 2\pi)\}$ where

$$\mathsf{C}(r) = \frac{1+r^2}{1-r^2}, \quad \mathsf{R}(r) = \frac{2r}{1-r^2}. \tag{8}$$

It should be noted that $\mathsf{R} \to \infty$ corresponds to $r \to 1$.

Scattering matrix

For brevity, we define

$$s(k) = \left|\det S(k)\right|,$$

and after throwing away constants which are irrelevant for convergence, we obtain the final form of the criterion (7), which is convenient for us and will be used afterwards:

$$\lim_{r\to 1}\int_0^{2\pi}\frac{\mathsf{R}(r)\ln(s(\mathsf{R}(r)e^{it}+i\mathsf{C}(r)))}{(\mathsf{R}(r)e^{it}+i\mathsf{C}(r)+i)^2}dt = 0. \tag{9}$$

Main theorem

Our main theorem is as follows.

Theorem 0.12 Main theorem. *The system of resonance states is complete on* $L^2(\Gamma_0)$ *if and only if every external vertex of* Γ *is unbalanced.*

Main theorem

To prove the theorem we analyze the algebraic system giving us the scattering matrix.
To construct the scattering matrix for the graph Γ we solve a series of scattering problems each of them corresponds to wave coming from one lead. In more details, the situation is as follows. The general solution at each edge and lead (say, e_q) has the form $\beta_q e^{-ikx} + \gamma_q e^{ikx}$. This is the solution of scattering problem for wave coming from $j-$th lead if the solution on $j-$th lead has the form $e^{-ikx} + s_{jj}e^{ikx}$ and on $p-$th lead, $p \neq j$, has the form $s_{pj}e^{ikx}$. Coefficients s_{pj} are entries of the scattering matrix. As for the solution at edges ($\beta_{qj}e^{-ikx} + \gamma_{qj}e^{ikx}$), we are not interesting in the values of β_{qj}, γ_{qj}. The linear system for determination of $s_{pj}, \beta_{qj}, \gamma_{qj}$ is given by the Dirichlet condition at boundary vertices, continuity at each non-boundary vertex, the Kirchhoff condition (10) at each non-boundary vertex. We have separate system for each j, i.e. the number of systems coincides with the number of semi-infinite leads of Γ.
We consider these systems in balanced and unbalanced cases.

Main theorem

The second stage is estimation of the integral from the completeness criterion.
The integration curve is divided into several parts. The first part is that inside a strip $0 < y < \delta$. Here we take into account that at the real axis $(y = 0)$ one has $s(k) = 1$.
The second part of the integral is related to the singularities, i.e., the roots of $s(k)$ (resonances). We estimate the integral in small neighborhoods of the singularities and at the rest of the curve.
Thus, the procedure of estimation is as follows, e.g., for unbalanced case. Choose δ_1', δ_1 to separate the root (or roots) of $s(k)$. If $t_0 - \delta_1 > 0$ then consider $(0, t_0 - \delta_1]$ separately (for the second semi-circle $\pi \le t < 2\pi$ the consideration is analogous). For this part of the curve with small t (i.e. small y), the estimate of the integral is $O(1/\sqrt{\mathsf{R}})$. For the part of the curve outside these intervals, the estimate of the integral is $O(1/\mathsf{R})$. Consequently, the full integral is estimated as $O(1/\sqrt{\mathsf{R}})$, i.e., the integral tends to zero if $\mathsf{R} \to \infty$. In accordance with the completeness criterion we have the completeness in this case.
The analogous procedure for the case when there is a balanced vertex gives us another result, non-zero limit, due to the exponential factor.

Landau operator

Consider the case of the Landau operator (one-dimensional Schrödinger operator with a magnetic field: $H = -(\frac{d}{dx} + iA_j)$, A_j is the tangent component of the vector potential corresponding to the magnetic field for edge e_j. We assume the Kirchhoff condition at the internal vertices:

$$\sum_{e, v \in e} (-1)^{\kappa(e(v))} \mathcal{D}\psi = 0, \qquad (10)$$

where $\kappa(e(v)) = 0$ for outgoing edge/lead e and $\kappa(e(v)) = 1$ for incoming edge e (we assumed earlier that all leads are outgoing), $\mathcal{D}$ is the "magnetic derivative".
Exner and Lipovsky (P. Exner, J. Lipovsky. Non-Weyl resonance asymptotics for quantum graphs in a magnetic field. Phys. Lett. A375 (2011), 805-807) proved that the magnetic field does not change the situation with Weyl/non-Weyl asymptotics of resonances in the case of general coupling condition. We proved that the same take place in respect to completeness for the Kirchhoff coupling condition.

Thank you for your attention!