Perturbations of manifolds and spectral convergence

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Differential Operators on Graphs and Waveguides – Graz

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Motivation: perturbed manifolds and convergence of Laplacians

- X Riemannian manifold (or subset of $\mathbb{R}^n$ or waveguide ...)
- "wild perturbations" (name by Rauch-Taylor [RT75]):
  - remove obstacles B_ε, e.g. many small balls, $X_\varepsilon := X \setminus B_\varepsilon$
  - add many small handles, resulting manifold X_ε
    (neither subset nor superset of X!)
- Question: Convergence of (Neumann/Dirichlet) Laplacian on X_ε? To what limit?
- Here mostly free limit: Laplacian on X
  (we call obstacles B_ε with $\Delta_{X_\varepsilon}^{(\mathrm{Neu/Dir})} \to \Delta_X$ fading)
- Another question: Now to define norm resolvent convergence if spaces change?

Interlude: What is a good generalisation of norm resolvent convergence?

$\Delta_\varepsilon \ge 0$ in Hilbert space $\mathscr{H}_\varepsilon$ for all $\varepsilon \ge 0$

Definition (Generalised norm resolvent convergence, P:06, P:12)

$\Delta_\varepsilon \xrightarrow{\text{gnrs}} \Delta_0$:⇔ there exist $J = J_\varepsilon \colon \mathscr{H}_0 \to \mathscr{H}_\varepsilon$ bdd. and $\delta_\varepsilon \to 0$ such that

$$\|(\mathrm{id}_0 - J^* J)R_0\| \le \delta_\varepsilon, \qquad \|(\mathrm{id}_\varepsilon - JJ^*)R_\varepsilon\| \le \delta_\varepsilon, \tag{1}$$

$$\|R_\varepsilon J - JR_0\| \le \delta_\varepsilon. \tag{2}$$

If (1)–(2) hold for some J, we call Δ_ε, Δ_0 δ_ε-quasi-unitary equivalent.

Generalisation of standard norm resolvent convergence

- $\delta_\varepsilon = 0$ for (1): J unitary; w.l.o.g. $\mathscr{H}_0 = \mathscr{H}_\varepsilon$, $J = \mathrm{id}$.
  Then (2) is $\|R_\varepsilon - R_0\| \to 0$

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Generalisation of unitary equivalence

- $\delta_\varepsilon = 0$ for (1)–(2): J unitary and Δ_ε, Δ_0 unititarily equivalent

Consequences of generalised norm resolvent convergence

Definition (Generalised norm resolvent convergence, P:06, P:12)

$\Delta_\varepsilon \xrightarrow{\text{gnrs}} \Delta_0$:⇔ there exist $J\colon \mathscr{H}_0 \to \mathscr{H}_\varepsilon$ bdd. and $\delta_\varepsilon \to 0$ such that
(1) $\|(\mathrm{id}_0 - J^*J)R_0\| \le \delta_\varepsilon$, $\|(\mathrm{id}_\varepsilon - JJ^*)R_\varepsilon\| \le \delta_\varepsilon$, (2) $\|R_\varepsilon J - JR_0\| \le \delta_\varepsilon$.

Theorem (P:06, P:12)

If $\Delta_\varepsilon \xrightarrow{gnrs} \Delta_0$, we have e.g.

- $\|\varphi(\Delta_\varepsilon) - J\varphi(\Delta_0)J^*\| \le C_\varphi \delta_\varepsilon$ *(e.g. $\varphi_t(\lambda) = \mathrm{e}^{-t\lambda}$, $\varphi = \mathbb{1}_I$)*
- $\sigma(\Delta_\varepsilon) \to \sigma(\Delta_0)$ *on compact intervals [also for discrete and essential spectrum], convergence of eigenfunctions (even in energy norm)*

In particular: no spectral pollution, no spurious eigenvalues:

$$\lambda_0 \in \sigma(\Delta_0) \iff \exists (\lambda_\varepsilon)_\varepsilon \colon \lambda_\varepsilon \in \sigma(\Delta_\varepsilon), \lambda_\varepsilon \to \lambda_0$$

Cannot have generalised norm resolvent convergence for compact spaces approximating a non-compact one (as essential spectra converge)

Fading Neumann obstacles

General assumption

X complete Riemannian manifold with Laplacian $\Delta_0 := \Delta_X$ of bounded geometry (injectivity radius $\iota > 0$, Ricci curvature bounded from below).

Consequences:

- $\rightsquigarrow$ Small balls look Euclidean!
- $\exists C_{\mathrm{ell}} > 0 \; \forall f \in \mathsf{H}^2(X)$: $\|f\|_{\mathsf{H}^2(X)} \le C_{\mathrm{ell}} \|(\Delta_X + 1)f\|_{\mathsf{L}_2(X)}$

Definition (Neumann fading obstacles)

B_ε are Neumann fading obstacles iff $B_\varepsilon \subset X$, $\delta_\varepsilon \to 0$,

$$\|f\|_{\mathsf{L}_2(B_\varepsilon)} \le \delta_\varepsilon \|f\|_{\mathsf{H}^1(X)}, \tag{4}$$

$$\exists E_\varepsilon \colon \mathsf{H}^1(X_\varepsilon) \to \mathsf{H}^1(X), E_\varepsilon u\restriction_{X_\varepsilon} = u\colon \; \|E_\varepsilon\|_{\mathsf{H}^1 \to \mathsf{H}^1} \le C_{\mathrm{ext}} \tag{5}$$

- (4) means: eigenfunctions non-concentrating on B_ε
- (5): $X_\varepsilon := X \setminus B_\varepsilon$ strongly connected (in homogenisation theory)

Neumann fading obstacles II

Theorem (Anné-P:18)

If $B_\varepsilon \subset X$ are Neumann fading obstacles then $\Delta^{\mathrm{Neu}}_{X_\varepsilon} \xrightarrow{gnrs} \Delta_X$.

Flavour of proof:

- Hilbert spaces: $\mathscr{H}_\varepsilon = \mathsf{L}_2(X_\varepsilon)$, $\mathscr{H}_0 := \mathsf{L}_2(X)$
- Energy forms: $\mathcal{E}_\varepsilon(u) = \int_{X_\varepsilon} |du|^2$, $\operatorname{dom} \mathcal{E}_\varepsilon = \mathsf{H}^1(X_\varepsilon)$,
  ($\rightsquigarrow$ Neumann Laplacian on X_ε)
  $\mathcal{E}_0(f) = \int_X |du|^2$, $\operatorname{dom} \mathcal{E}_0 = \mathsf{H}^1(X)$
- Identification operators: $J\colon \mathsf{L}_2(X_0) \to \mathsf{L}_2(X_\varepsilon)$, $Jf := f\restriction_{X_\varepsilon}$,
  hence $J^* u = \bar{u} := u \oplus 0$.

Then: $u = JJ^* u$ and $f - J^* Jf = f\restriction_{B_\varepsilon} \oplus 0$,
Hence:

$$\|f - J^* Jf\|^2 = \|f\|^2_{\mathsf{L}_2(B_\varepsilon)} \overset{(4)}{\leq} \delta_\varepsilon^2 \|f\|^2_{\mathsf{H}^1(X)} = \delta_\varepsilon^2 \|(\Delta_X + 1)^{1/2} f\|^2_{\mathsf{L}_2(X)},$$

(non-concentrating on B_ε) hence (1) fulfilled with δ_ε

Neumann fading obstacles III

Example:

For each $\varepsilon > 0$ let $\eta_\varepsilon > 0$ and $I_\varepsilon \subset X$ be an η_ε-separated set, i.e.,

$$x, y \in I_\varepsilon, \quad x \neq y \implies d(x, y) \geq 2\eta_\varepsilon$$

then $B_\varepsilon := B_{\eta_\varepsilon}(I_\varepsilon) = \bigcup_{x \in I_\varepsilon} B_{\eta_\varepsilon}(x)$ is a disjoint union of small balls.
We assume that

$$0 < \varepsilon \ll \eta_\varepsilon \ll 1, \qquad \text{i.e.,} \qquad \eta_\varepsilon \to 0, \quad \frac{\varepsilon}{\eta_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0$$

(e.g. $\eta_\varepsilon = \varepsilon^\alpha$, $\alpha \in (0,1)$).

Proposition (Anné-P:18)

B_ε are Neumann fading obstacles with $\delta_\varepsilon = \mathrm{O}(\varepsilon/\eta_\varepsilon)$ ($n \geq 3$).

Corollary (Anné-P:18)

If B_ε are disjoint balls as above then $\Delta_{X_\varepsilon}^{\mathrm{Neu}} \xrightarrow{gnrs} \Delta_X$.

Fading Dirichlet obstacles

Definition (Dirichlet fading obstacles)

B_ε are Dirichlet fading obstacles iff $B_\varepsilon \subset X$, $\delta_\varepsilon \to 0$, $\exists \chi_\varepsilon \colon X \to [0,1]$, $B_\varepsilon^+ := \{ x \in X \mid \chi_\varepsilon(x) = 1 \}$:

$$\|f\|_{\mathsf{L}_2(B_\varepsilon^+)} \le \delta_\varepsilon \|f\|_{\mathsf{H}^1(X)}, \tag{6}$$

$$\mathsf{H}^2(X) \to \mathsf{L}_2(T^*(B_\varepsilon^+)), \quad f \mapsto f d\chi_\varepsilon \restriction_{B_\varepsilon^+} \tag{7}$$

Theorem ([Anné-P:18])

Assume $B_\varepsilon \subset X$ are Dirichlet fading obstacles then Then $\Delta_{X_\varepsilon}^{\mathrm{Dir}} \xrightarrow{gnrs} \Delta_X$.

Corollary ([Anné-P:18])

Assume $B_\varepsilon := B_{\eta_\varepsilon}(I_\varepsilon) = \bigcup_{x \in I_\varepsilon} B_{\eta_\varepsilon}(x)$ is ε^α-separated, $\alpha \in (0, (m-2)/m)$, then $\Delta_{X_\varepsilon}^{\mathrm{Dir}} \xrightarrow{gnrs} \Delta_X$.

Meaning of the fading conditions in Neu/Dir case

Assume X is compact then by the η_ε-separation the number of balls $N_\varepsilon = |I_\varepsilon|$ is finite.
Moreover if $\eta_\varepsilon = \varepsilon^\alpha$, then $\operatorname{vol} B_{\eta_\varepsilon} \sim \varepsilon^{\alpha m}$ and $N_\varepsilon \varepsilon^{\alpha m} \lesssim 1$, hence

$$\text{(number of balls)} \quad N_\varepsilon \lesssim \varepsilon^{-\alpha m}.$$

When do the obstacles fade away ...?

- Neumann: $0 < \alpha < 1$: number of balls less than ε^{-m}, or

$$N_\varepsilon \varepsilon^m \cong \varepsilon^{(1-\alpha)m} \to 0 \qquad \text{(volume of obstacles)}.$$

- Dirichlet ($m \geq 3$): $0 < \alpha < \dfrac{m-2}{m}$: number of balls less than $\varepsilon^{-(m-2)}$, or

$$N_\varepsilon \varepsilon^{m-2} \to 0 \qquad \text{(capacity of obstacles)}.$$

Homogenisation — the critical case

Consider the case $\alpha = (m-2)/m$:
Let $X_\varepsilon = X \setminus B_\varepsilon$, B_ε union of balls (little obstacles) $D_\varepsilon = \varepsilon D$ of radius ε in a grid of length $\eta_\varepsilon = \varepsilon^\alpha$
Let $q = \lim_{\varepsilon \to 0} \frac{\mathrm{cap}(D_\varepsilon)}{\varepsilon^{\alpha m}}$ exist.

Theorem (Khrabustovskiy-Post:18)

We have $\Delta_{X_\varepsilon}^{\mathrm{Dir}} \xrightarrow{gnrs} \Delta_X + q$.

Typical results only include strong resolvent convergence.

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