Hot spots of quantum graphs

Jonathan Rohleder

Ongoing joint work with James Kennedy (Lisbon)

Graz, February 27, 2019

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For an initial condition u_0 the diffusion of heat in Ω described by

$$u(t,x) = e^{t\Delta_\Omega^N} u_0(x), \qquad x \in \Omega,\ t > 0.$$

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Let $0 = \mu_1 \leq \mu_2 \leq \ldots$ eigenvalues, $\psi_1, \psi_2, \ldots$ eigenvectors of $-\Delta_\Omega^N$. Then

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Let $\Omega \subset \mathbb{R}^d$ be a bounded, smooth domain and ψ_2 any eigenfunction associated with the second Neumann eigenvalue μ_2. Then

$$\max_{x \in \overline{\Omega}} \psi_2(x), \quad \min_{x \in \overline{\Omega}} \psi_2(x)$$

are achieved (only) on $\partial\Omega$.

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- ... is open for general *convex* domains, even in two dimensions.

Quantum Graphs

Now $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ connected, compact metric graph:

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- $\mu_2 > 0$ and its eigenfunction(s) ψ_2 change sign in $\mathcal{G}$.

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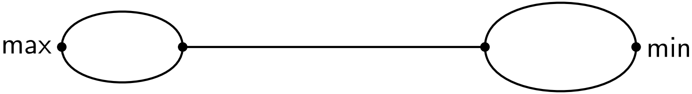
- Does M realise the diameter of $\mathcal{G}$, i.e., can one find $x, y \in M$ s.t. $\operatorname{dist}(x, y) = \operatorname{diam}\mathcal{G}$? (Or at least $\operatorname{dist}(x, y) \approx \operatorname{diam}\mathcal{G}$?)

Some (counter-)examples

M need not have anything to do with $\partial\mathcal{G}$:



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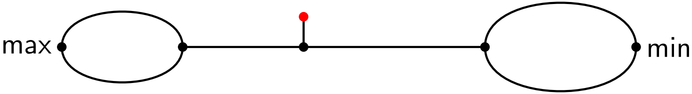

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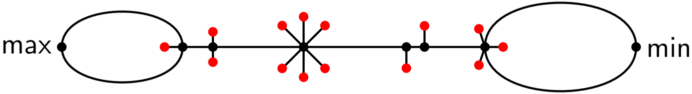

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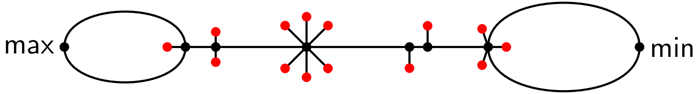

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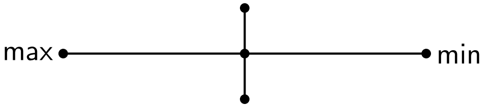

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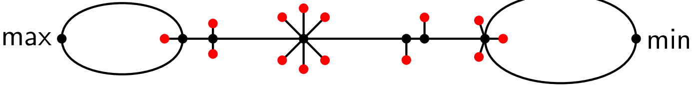

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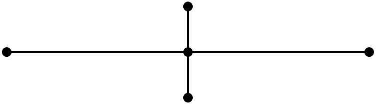

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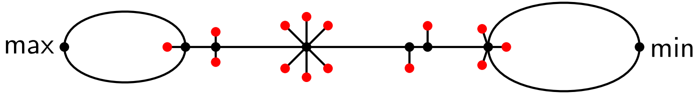

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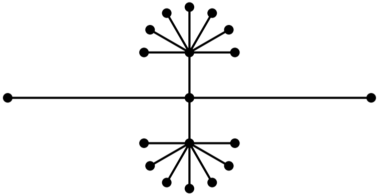

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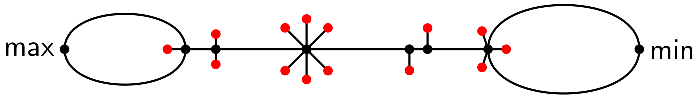

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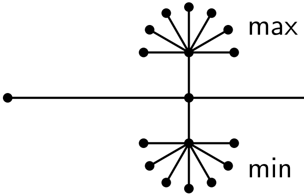

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For the discrete Laplacian on trees: Fiedler 1975, Evans 2011, Levèvre 2013, Gernandt and Pade 2018

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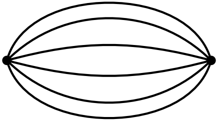

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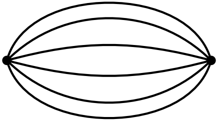

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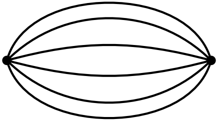

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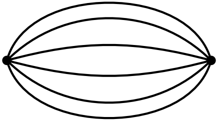

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Thus for most graphs there are two "distinguished" points where the heat (or cold) is asymptotically most concentrated.

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