Local energy decay for the wave equation in a dissipative wave guide

Julien Royer

Institut de Mathématiques de Toulouse - France

Differential Operators on Graphs and Waveguides

TU Graz

February 25, 2018

Let u be the solution in $\mathbb{R}^d$ of the free wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

We have conservation of the energy :

$$\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 = \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 .$$

If u_0 and u_1 are compactly supported and $\chi \in C_0^\infty(\mathbb{R}^d)$:

- $d \geqslant 3$ odd: propagation at speed 1 (Huygens' principle)

$$\|\chi(x)\nabla u(t,x)\|^2 + \|\chi(x)\partial_t u(t,x)\|^2 = 0 \qquad \text{for } t \text{ large enough.}$$

- d even: propagation at speed $\leqslant 1$

$$\|\chi(x)\nabla u(t,x)\|^2 + \|\chi(x)\partial_t u(t,x)\|^2 \lesssim t^{-2d}\left(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2\right).$$

- Wave equation in an exterior domain or for a Laplace-Beltrami operator (given by the refraction index)

- Uniform decay for the local energy under the non-trapping condition on the classical flow (assumption for high frequencies).

$$\|\chi(x)\nabla u(t,x)\|^2 + \|\chi(x)\partial_t u(t,x)\|^2 \lesssim \begin{Bmatrix} e^{-\gamma t} \\ t^{-2d} \end{Bmatrix} \left(\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 \right).$$

(Lax-Morawetz-Philipps '63, Ralston '69, Morawetz-Ralston-Strauss '77, Bony-Häfner '10, Bouclet '11, Bouclet-Burq '19)

- Without non-trapping: Logarithmic decay with loss of regularity (Burq '98):

$$\|\chi(x)\nabla u(t,x)\|^2 + \|\chi(x)\partial_t u(t,x)\|^2 \lesssim \ln(2+t)^{-2k} \left(\|u_0\|_{H^{k+1}}^2 + \|u_1\|_{H^k}^2 \right).$$

- Various intermediate settings ...

Stabilization in a bounded domain

On $\Omega \subset \mathbb{R}^d$ bounded, we consider the damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + a\partial_t u = 0 \quad \text{on } \Omega, \\ + \text{ Dirichlet or Neumann at the boundary,} \\ + \text{ initial conditions.} \end{cases}$$

- $a(x) \geqslant 0$ is the absorption index
- The global energy decays: for $t_1 \leqslant t_2$

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2} \int_\Omega a(x) \left|\partial_t u(s,x)\right|^2 \, dx \, ds \leqslant 0.$$

- Uniform decay under the Geometric Control Condition.

We can also consider damping at the boundary:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{on } \Omega, \\ \partial_\nu u + a\partial_t u = 0 \quad \text{on } \partial\Omega, \\ + \text{ initial conditions.} \end{cases}$$

See Rauch-Taylor '74, Bardos-Lebeau-Rauch '92, Lebeau '96,...

Damped wave equation on unbounded domains

For the damped wave equation in an unbounded domain, the expected necessary and sufficient condition for uniform decay of the local energy is:

Each bounded classical trajectory goes through the damping region.

- Aloui-Khenissi '02, Khenissi '03: damped wave equation in an exterior domain (compactly supported absorption index).
- Bouclet-R. '14, R. '16: asymptotically free damped wave equation.

$\rightsquigarrow$ We recover the same rates of decay as for the undamped case under the non-trapping condition.

- For the damped wave equation in an unbounded domain, the expected necessary and sufficient condition for uniform decay of the local energy is:

  Each bounded classical trajectory goes through the damping region.

- Absorption index effective at infinity (for instance $a(x) \to 1$ as $|x| \to \infty$)
  Model case:

$$\partial_t^2 u - \Delta u + \partial_t u = 0$$

$\rightsquigarrow$ The local energy decays like t^{-d-2}.
This can be slower than for the undamped wave. For low frequencies, the solution of the damped wave equation behaves like a solution of the heat equation

$$-\Delta v + \partial_t v = 0.$$

  - Matsumura '76 (decay estimates)
  - Orive-Pozato-Zuazua '01 (periodic medium, constant damping)
  - Marcati-Nishihara '03, Nishihara '03, Hosono-Ogawa '04, Narazaki '04 (diffusive phenomenon)
  - Ikehata '02, Aloui-Ibrahim-Khenissi '15 (exterior domains)
  - + time dependant damping + semi-linear equation + abstract results + ...

Damped wave equation with strong damping

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1. On a wave guide $\Omega = \mathbb{R}^d \times \omega$ with dissipation at the boundary:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u + a\partial_t u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

2. On a wave guide with dissipation at infinity (with M. Malloug, Sousse):

$$\begin{cases} \partial_t^2 u - \Delta u + a\partial_t u = 0 & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

where $a(x,y) \to 1$ as $|x| \to \infty$.

3. On an asymptotically periodic medium (with R. Joly, Grenoble)

$$\partial_t^2 u + Pu + a(x)\partial_t u = 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^d,$$

where

$$P = -\operatorname{div} G(x)\nabla, \quad G(x) = G_{\text{per.}}(x) + G_{\to 0}(x), \quad a(x) = a_{\text{per.}}(x) + a_{\to 0}(x)$$

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The energy is non-increasing (here the absorption index a is a positive constant):

$$\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 \right) = -2 \int_{\partial\Omega} a \left|\partial_t u(t)\right|^2 \, dt \leqslant 0.$$

Theorem (Local Energy Decay)

Let $\delta > \frac{d}{2} + 1$. Then there exists $C \geqslant 0$ such that for $u_0 \in H^{1,\delta}(\Omega)$ and $u_1 \in L^{2,\delta}(\Omega)$ we have for all $t \geqslant 0$

$$\left\| \langle x \rangle^{-\delta} \nabla u(t) \right\|_{L^2(\Omega)}^2 + \left\| \langle x \rangle^{-\delta} \partial_t u(t) \right\|_{L^2(\Omega)}^2 \leqslant C \langle t \rangle^{-d-2} \left(\left\| \langle x \rangle^{\delta} \nabla u_0 \right\|_{L^2(\Omega)}^2 + \left\| \langle x \rangle^{\delta} u_1 \right\|_{L^2(\Omega)}^2 \right).$$

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Diffusive asymptotics

We consider the solution v for the following heat equation on $\mathbb{R}^d$:

$$\begin{cases} a\kappa\partial_t v - \Delta_x v = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ v|_{t=0} = v_0(u_0, u_1) & \text{on } \mathbb{R}^d, \end{cases}$$

where

$$\kappa = \frac{|\partial\omega|}{|\omega|}.$$

We set $v(t; x, y) = v(t; x)$ ($x \in \mathbb{R}^d$, $y \in \omega$).

Theorem (Comparison with the Heat Equation)

Let $\delta > \frac{d}{2} + 1$. Then there exists $C \geqslant 0$ such that for $u_0 \in H^{1,\delta}(\Omega)$ and $u_1 \in L^{2,\delta}(\Omega)$ we have for all $t \geqslant 0$

$$\left\| \langle x \rangle^{-\delta} \nabla(u - v)(t) \right\|_{L^2(\Omega)}^2 + \left\| \langle x \rangle^{-\delta} \partial_t(u - v)(t) \right\|_{L^2(\Omega)}^2 \leqslant C \langle t \rangle^{-d-4} \left(\left\| \langle x \rangle^{\delta} \nabla u_0 \right\|_{L^2(\Omega)}^2 + \left\| \langle x \rangle^{\delta} u_1 \right\|_{L^2(\Omega)}^2 \right).$$

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- By the general dissipative theory (maximal dissipative operators, Hille-Yosida theorem, contractions semi-groups, etc.), we can check that our wave equation has a solution defined for all positive times.

- After a Fourier transform, the problem reduces to proving uniform estimates for the derivatives of the "resolvent"

$$R_a(z) = \left(H_{a\tau} - \tau^2\right)^{-1}, \quad \tau \in \mathbb{R}.$$

We have denoted by H_α the operator $-\Delta$ with domain

$$\mathcal{D}(H_\alpha) = \left\{u \in H^2(\Omega) \,:\, \partial_\nu u = i\alpha u \text{ on } \partial\Omega\right\}.$$

- Qu : how do we compute the derivative of $R_a(\tau)$ with respect to τ ?

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We can write $H_\alpha = -\Delta_x + T_\alpha$ where:

- $-\Delta_x$ is the usual Laplacian on $\mathbb{R}^d$. It is selfadjoint, its spectrum is $\mathbb{R}_+$.
- T_α is defined as H_α but on ω instead of Ω:

$$T_\alpha = -\Delta_\omega, \quad \mathcal{D}(T_\alpha) = \left\{ u \in H^2(\omega) \, : \, \partial_\nu u = i\alpha u \text{ on } \partial\omega \right\}.$$

  It has compact resolvent. Its spectrum is given by a sequence of eigenvalues $(\lambda_n(\alpha))$.

The spectrum of H_α is

$$\sigma(H_\alpha) = \sigma(T_\alpha) + \mathbb{R}_+.$$

- For $\alpha > 0$ this gives a sequence of half-lines under the real axis.

$\rightsquigarrow$ For $\tau > 0$ the operator $(H_{a\tau} - \tau^2)$ is boundedly invertible.

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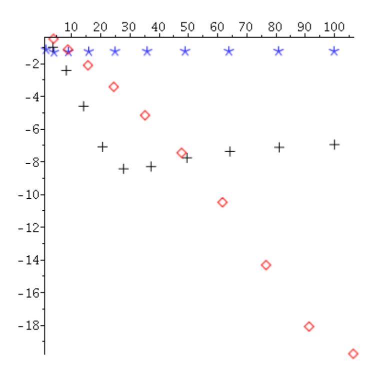
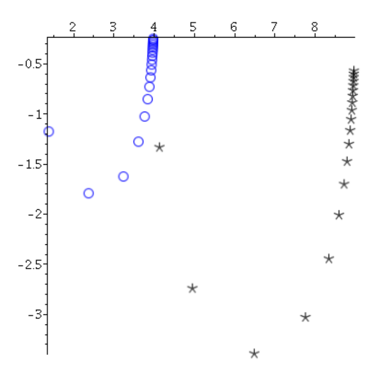
For high frequencies we first have to understand the behavior of $T_{a\tau}$ as $\tau \to \infty$.





For $\tau \gg 1$, the first eigenvalues of $T_{a\tau}$ are close to the real axis (close to Dirichlet eigenvalues). In particular

$$\text{dist}\left(\tau^2, \mathrm{Sp}(H_{a\tau})\right) \xrightarrow[\tau\to\infty]{} 0.$$

Contribution of low frequencies

- The first eigenvalue of T_0 is 0, it is a simple eigenvalue and $\varphi_0(0)$ is constant on ω.
- We have

$$\frac{d\lambda_0(\alpha)}{d\alpha}\Big|_{\alpha=0} = -i\kappa.$$

- For τ small, we have

$$\begin{aligned}\big(H_{a\tau}-\tau^2\big)^{-1} &= \big(-\Delta_x-\tau^2+\lambda_0(a\tau)\big)^{-1}\langle\cdot,\varphi_0(a\tau)\rangle\,\varphi_0(a\tau)+\text{rest}\\ &= \big(-\Delta_x-ia\kappa\tau\big)^{-1}P_\omega+\text{rest},\end{aligned}$$

where

$$P_\omega u(x) = \frac{1}{|\omega|}\int_\omega u(x,\cdot)$$

- $\big(-\Delta_x-ia\kappa\tau\big)^{-1}$ is the resolvent corresponding to the heat equation

$$a\kappa\partial_t v - \Delta_x v = 0,$$

and the rest will give a contribution of size $O(t^{-d-4})$ for the local energy.

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$$\begin{aligned}\left(H_{a\tau} - \tau^2\right)^{-1} &= \left(- \Delta_x - \tau^2 + \lambda_0(a\tau)\right)^{-1} \langle \cdot, \varphi_0(a\tau)\rangle \varphi_0(a\tau) + \text{rest}\\ &= \left(- \Delta_x - ia\kappa\tau\right)^{-1} P_\omega + \text{rest},\end{aligned}$$

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and the rest will give a contribution of size $O(t^{-d-4})$ for the local energy.

- The first eigenvalue of T_0 is 0, it is a simple eigenvalue and $\varphi_0(0)$ is constant on ω.
- We have

$$\frac{d\lambda_0(\alpha)}{d\alpha}\Big|_{\alpha=0} = -i\kappa.$$

- For τ small, we have

$$\begin{aligned}\left(H_{a\tau} - \tau^2\right)^{-1} &= \left(-\Delta_x - \tau^2 + \lambda_0(a\tau)\right)^{-1} \langle \cdot, \varphi_0(a\tau)\rangle \varphi_0(a\tau) + \text{rest} \\ &= \left(-\Delta_x - ia\kappa\tau\right)^{-1} P_\omega + \text{rest},\end{aligned}$$

where

$$P_\omega u(x) = \frac{1}{|\omega|}\int_\omega u(x,\cdot)$$

- $\left(-\Delta_x - ia\kappa\tau\right)^{-1}$ is the resolvent corresponding to the heat equation

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Contribution of low frequencies

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In dimension 1 we can check by explicit computation that

$$\mathrm{Re}(\lambda_k) \sim \tau^2 \quad \Rightarrow \quad \mathrm{Im}(\lambda_k) \lesssim -\tau.$$

This is in fact a general result (proved by semiclassical analysis on ω):

Proposition

There exist $\tau_0 \geqslant 1$, $\gamma > 0$ and $c \geqslant 0$ such that for $\tau \geqslant \tau_0$ and $\zeta \in \mathbb{C}$ which satisfy

$$\left|\mathrm{Re}(\zeta - \tau^2)\right| \leqslant \gamma\tau^2 \quad \textit{and} \quad \mathrm{Im}(\zeta) \geqslant -\gamma\tau$$

the resolvent $(T_{a\tau} - \zeta)^{-1}$ is well defined and we have

$$\left\|(T_{a\tau} - \zeta)^{-1}\right\|_{\mathcal{L}(L^2(\omega))} \leqslant \frac{c}{\tau}.$$

If we restrict the frequency in the x-directions, we get the following estimate:

Proposition

Let τ_0 and γ be as above. If χ is supported in $]-\gamma,\gamma[$ then there exists $c \geqslant 0$ such that for $\tau \geqslant \tau_0$ we have

$$\left\|\chi(-\Delta_x/\tau^2) R_a(\tau)\right\|_{\mathcal{L}(L^2(\Omega))} \leqslant \frac{c}{\tau}.$$

Proposition

Let τ_0 be as above. Let $\delta > \frac{1}{2}$. Then there exists $c \geqslant 0$ such that for $\tau \geqslant \tau_0$ we have

$$\left\| \langle x \rangle^{-\delta} (1-\chi)(-\Delta_x/\tau^2) R_a(\tau) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\Omega))} \leqslant \frac{c}{\tau}.$$

- We use an escape function as can be done in the Euclidean space.
- We use pseudo-differential operators only in the x-directions.
- On the Euclidean space, see Robert-Tamura '87.

Theorem (High frequency estimates)

Let $\delta > \frac{1}{2}$. Then there exists $c \geqslant 0$ such that for $\tau \geqslant \tau_0$ we have

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Then we need estimates for the derivatives of this resolvent...

Contribution of high longitudinal frequencies

Proposition

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Then we need estimates for the derivatives of this resolvent. . .

For z small, only the contribution of $\lambda_0(a\tau)$ should play a role.
If H_α is selfadjoint we can write

$$(H_\alpha - \zeta)^{-1} u = \sum_{n\in\mathbb{N}} (-\Delta_x + \lambda_\alpha(n) - \zeta)^{-1} u_n(x) \otimes \varphi_n(\alpha)$$

where $T_\alpha \varphi_n(\alpha) = \lambda_n(\alpha)\varphi_n(\alpha)$, $u(x,y) = \sum u_n(x) \otimes \varphi_n(\alpha; y)$. Then

$$\left\|(H_\alpha - \zeta)^{-1} u\right\|^2_{L^2(\Omega)} = \sum_{n\in\mathbb{N}} \left\|(-\Delta_x + \lambda_\alpha(n) - \zeta)^{-1} u_n(x)\right\|^2_{L^2(\mathbb{R}^d)}$$

- We consider $\mathcal{G}$ of the form

$$\mathcal{G} = \{\zeta \in \mathbb{C} : \operatorname{Re}(\zeta) < R_1, |\operatorname{Im}(\zeta)| < R_2\}.$$

  and the projection

$$P_{\mathcal{G}} = -\frac{1}{2i\pi} \int_{\partial\mathcal{G}} (T_\alpha - \sigma)^{-1}\, d\sigma \quad \in \mathcal{L}(L^2(\omega)).$$

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$$(T_\alpha - \sigma)^{-1}(-\Delta_x - \zeta + \sigma)^{-1} = (H_\alpha - \zeta)^{-1}(-\Delta_x - \zeta + \sigma)^{-1} + (H_\alpha - \zeta)^{-1}(T_\alpha - \sigma)^{-1}.$$

- After integration over $\sigma \in \partial\mathcal{G}$ we get

$$(H_\alpha - \zeta)^{-1} = (H_\alpha - \zeta)^{-1} P_{\mathcal{G}} + \mathcal{B}_\alpha(\zeta),$$

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Separation of the spectrum

For z small, only the contribution of $\lambda_0(a\tau)$ should play a role.
If H_α is selfadjoint we can write

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where $T_\alpha \varphi_n(\alpha) = \lambda_n(\alpha)\varphi_n(\alpha)$, $u(x,y) = \sum u_n(x) \otimes \varphi_n(\alpha; y)$. Then

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- After integration over $\sigma \in \partial\mathcal{G}$ we get

$$(H_\alpha - \zeta)^{-1} = (H_\alpha - \zeta)^{-1} P_{\mathcal{G}} + \mathcal{B}_\alpha(\zeta),$$

  where $\mathcal{B}_\alpha$ is holomorphic in $\mathcal{G}$.

We recall the resolvent identity

$$R_a(\tau)(T_{a\tau}-\sigma)^{-1} = (T_{a\tau}-\sigma)^{-1}(-\Delta_x-\tau^2+\sigma)^{-1} - R_a(\tau)(-\Delta_x-\tau^2+\sigma)^{-1}.$$

We compose with $\chi(-\Delta/\tau^2)$ and integrate over σ on the boundary of

$$\mathcal{G}_\tau = \left\{\zeta \in \mathbb{C} : \left|\mathrm{Re}(\zeta) - \tau^2\right| \leqslant \gamma\tau^2, |\mathrm{Im}(\zeta)| \leqslant \gamma\tau\right\}.$$

Since $\mathcal{G}_\tau \cap \sigma(T_{a\tau}) = \emptyset$ the contribution of the left-hand side vanishes. The last term gives $R_a(\tau)\chi(-\Delta_x/\tau^2)$. Let E be the spectral measure associated to $-\Delta_x$. We have

$$\begin{aligned}
\mathcal{R}(\tau) &= \int_{\partial\mathcal{G}_\tau} (T_{a\tau}-\sigma)^{-1}\chi(-\Delta_x/\tau^2)\big(-\Delta_x-\tau^2+\sigma\big)^{-1}\,d\sigma\\
&= \int_{\partial\mathcal{G}_\tau} (T_{a\tau}-\sigma)^{-1}\left(\int_0^{+\infty} \frac{\chi(\Xi/\tau^2)}{\Xi-\tau^2+\sigma}\,dE(\Xi)\right)d\sigma\\
&= \int_0^{\gamma\tau^2} \chi(\Xi/\tau^2)\left(\int_{\partial\mathcal{G}_\tau} \frac{(T_{a\tau}-\sigma)^{-1}}{\Xi-\tau^2+\sigma}\,d\sigma\right)dE(\Xi)\\
&= \int_0^{\gamma\tau^2} \chi(\Xi/\tau^2)\left(\int_{\partial\mathcal{G}_{\tau,\Xi}} \frac{(T_{a\tau}-\sigma)^{-1}}{\Xi-\tau^2+\sigma}\,d\sigma\right)dE(\Xi)
\end{aligned}$$

where

$$\mathcal{G}_{\tau,\Xi} = \left\{\zeta \in \mathcal{G}_\tau : \left|\mathrm{Re}(\zeta) - \tau^2 + \Xi\right| \leqslant \gamma\tau\right\}.$$