Schrödinger operators with guided potentials on periodic graphs

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Periodic graphs

Let $G = (\mathcal{V}, \mathcal{E}) \subset \mathbb{R}^2$ be a connected infinite graph, $\mathcal{V}$ is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges.

Let $\Gamma \subset \mathbb{R}^2$ be a lattice with a basis a_1, a_2, i.e.,

$$\Gamma = \big\{ a \in \mathbb{R}^2 : a = n_1 a_1 + n_2 a_2, \quad n_1, n_2 \in \mathbb{Z} \big\},$$

$$\Omega = \big\{ x \in \mathbb{R}^2 : x = t_1 a_1 + t_2 a_2, \quad t_1, t_2 \in [0, 1) \big\}$$

be the *fundamental cell* of the lattice Γ.

We consider *locally finite* Γ*-periodic graphs* G, i.e., graphs satisfying the following conditions:

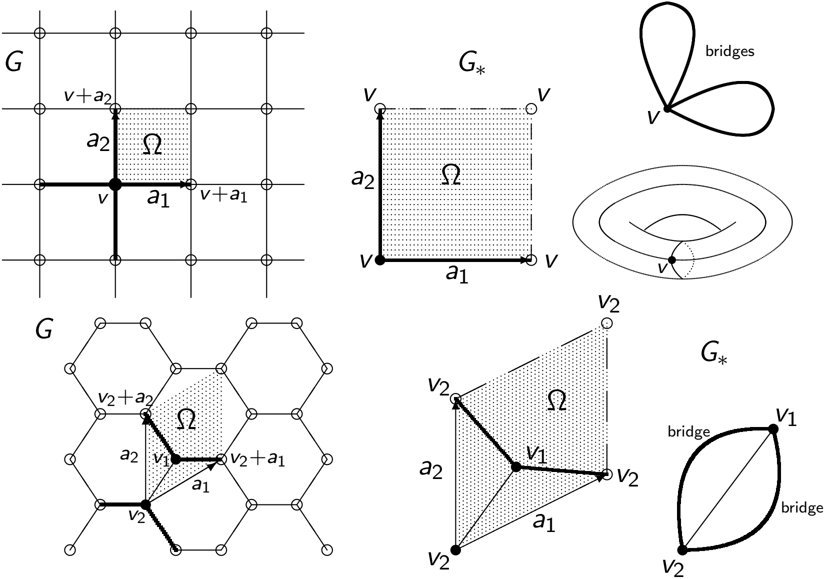
1) $G = G + a$ for any $a \in \Gamma$;

2) *the fundamental graph* $G_* = G/\Gamma$ is finite.

The vectors a_1, a_2 are called *the periods* of G.

$G_* = (\mathcal{V}_*, \mathcal{E}_*)$ is a graph on the 2-dimensional torus $\mathbb{R}^2/\Gamma$ and has the vertex set $\mathcal{V}_* = \mathcal{V}/\Gamma$ and the set $\mathcal{E}_* = \mathcal{E}/\Gamma$ of unoriented edges.

Examples of periodic graphs and fundamental graphs



Bridges are edges of G connecting the vertices from Ω (black points) with the vertices outside Ω (white points).

Schrödinger operators on periodic graphs

Let $\ell^2(\mathcal{V})$ be the Hilbert space of all square summable functions $f : \mathcal{V} \to \mathbb{C}$, equipped with the norm

$$\|f\|^2_{\ell^2(\mathcal{V})} = \sum_{v \in \mathcal{V}} |f(v)|^2 < \infty. \tag{1}$$

We consider the discrete Schrödinger operator H_0 on $f \in \ell^2(\mathcal{V})$:

$$H_0 = \Delta + W,$$

where Δ is the discrete combinatorial Laplace operator given by

$$(\Delta f)(v) = \sum_{(v,u) \in \mathcal{E}} \big(f(v) - f(u)\big), \quad f \in \ell^2(\mathcal{V}), \quad v \in \mathcal{V}. \tag{2}$$

The potential W is real-valued and Γ-periodic, i.e.,

$$(Wf)(v) = W(v)f(v), \quad W(v+a) = W(v), \quad \forall (v,a) \in \mathcal{V} \times \Gamma.$$

H_0 is self-adjoint and bounded.
We consider H_0 as an *unperturbed operator*.

The spectrum of the unperturbed operator H_0

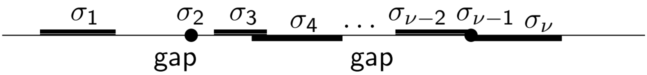
The spectrum $\sigma(H_0)$ of the Schrödinger operator $H_0 = \Delta + W$ with a periodic potential W on periodic graphs is a union of ν bands $\sigma_n = \sigma_n(H_0)$:

$$\sigma(H_0) = \bigcup_{n=1}^{\nu} \sigma_n(H_0) = \sigma_{ac}(H_0) \cup \sigma_{fb}(H_0),$$

where $\nu = \#\mathcal{V}_*$ is the number of vertices of the fundamental graph $G_* = (\mathcal{V}_*, \mathcal{E}_*)$.

The absolutely continuous spectrum $\sigma_{ac}(H_0)$ consists of non-degenerate bands $\sigma_n(H_0)$;

$\sigma_{fb}(H_0)$ is the set of all flat bands (eigenvalues of infinite multiplicity).



Schrödinger operators with guided potentials

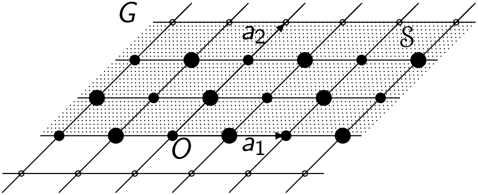
We consider a family of *guided Schrödinger operators* H_t, $t > 0$, on the periodic graph $G = (\mathcal{V}, \mathcal{E})$ given by

$$H_t = H_0 + tQ, \qquad \big(Qf\big)(v) = Q(v)f(v), \qquad f \in \ell^2(\mathcal{V}), \quad (3)$$

where $H_0 = \Delta + W$ is the unperturbed Schrödinger operator with a periodic potential W, and $Q \geqslant 0$ is a **guided potential** defined by

1) $\operatorname{supp} Q \subset \mathcal{S} = \mathbb{R} \times [0, 1)$ *(w.r.t. the periods a_1, a_2 of G);*
2) $Q(v + a_1) = Q(v)$ *for all $v \in \mathcal{V}$.*

The guided potential Q is periodic in the direction a_1 and finitely supported in the direction a_2.



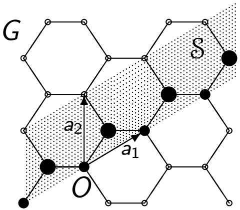


Figure: The strip $\mathcal{S} = \mathbb{R} \times [0, 1)$ is shaded. The black vertices are the support of the guided potential Q.

Spectrum of the guided Schrödinger operator

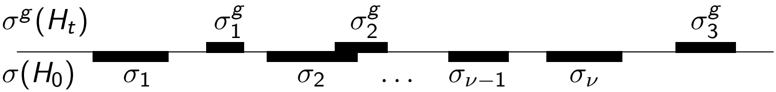
The spectrum of the guided Schrödinger operator $H_t = H_0 + tQ$ on periodic graphs has the form

$$\sigma(H_t) = \sigma(H_0) \cup \sigma^g(H_t), \qquad \sigma^g(H_t) = \bigcup_m \sigma^g_m(H_t),$$

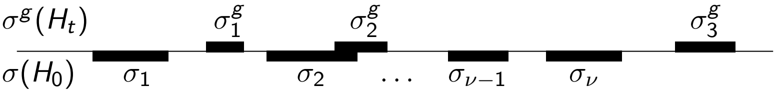
where $\sigma(H_0)$ is the spectrum of the unperturbed Schrödinger operator H_0,

$\sigma^g(H_t)$ is the additional *guided* spectrum which is a union of a finite number of the guided bands $\sigma^g_m \equiv \sigma^g_m(H_t)$.

$\sigma^g(H_t)$ may partly lie above the spectrum of H_0, on the spectrum of H_0 and in the gaps of H_0:



Korotyaev, E.; Saburova, N. Schrödinger operators with guided potentials on periodic graphs, Proc. Amer. Math. Soc., 145 (2017), no. 11, 4869–4883.



In that paper we considered the guided spectrum $\sigma^g(H_t)$ **above the spectrum of** H_0:

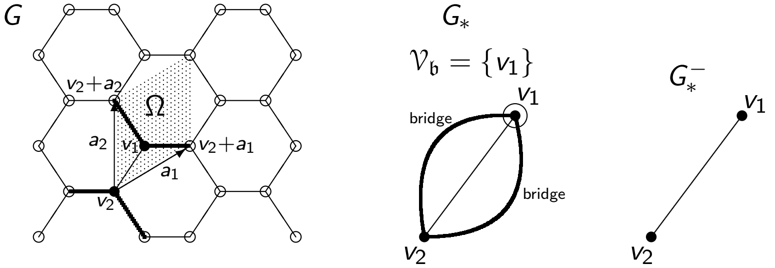
- we estimated the position of the guided bands and their length in terms of geometric parameters of the graph;
- we determined the asymptotics of the guided bands for large guided potentials;
- we showed that the possible number of the guided bands, their length and position can be rather arbitrary for some specific potentials.

But there are **no results** about the guided spectrum $\sigma^g(H_t)$ **in gaps of** H_0.

Auxiliary operators on finite graphs

Let G_*^- be the graph obtained from the fundamental graph $G_* = (\mathcal{V}_*, \mathcal{E}_*)$ by deleting all bridges.

Let $\mathcal{V}_\mathfrak{b} \subset \mathcal{V}_*$ be a set of vertices of G_* such that each bridge of G_* is incident to at least one vertex from $\mathcal{V}_\mathfrak{b}$. The set $\mathcal{V}_\mathfrak{b}$ is not uniquely defined.



For each $t \geqslant 0$ we consider two operators:

- $H_t^- = \Delta^- + W + tQ$ is the Schrödinger operator on G_*^-, where Δ^- is the Laplacian on G_*^-;
- H_t^+ is the Schrödinger operator $H_t = \Delta + W + tQ$ on G_* with the Dirichlet boundary conditions $f\restriction_{\mathcal{V}_\mathfrak{b}} = 0$.

Localization of spectral bands of H_0

- $H_t^- = \Delta^- + W + tQ$ is the Schrödinger operator on G_*^-;
- H_t^+ is the Schrödinger operator $H_t = \Delta + W + tQ$ on $G_* = (\mathcal{V}_*, \mathcal{E}_*)$ with the Dirichlet boundary conditions $f\restriction_{\mathcal{V}_\mathfrak{b}} = 0$.

The eigenvalues of H_t^- and H_t^+:

$$\mu_1^-(t) \leqslant \ldots \leqslant \mu_\nu^-(t), \qquad \nu = \#\mathcal{V}_*,$$

$$\mu_1^+(t) \leqslant \ldots \leqslant \mu_{\nu-r}^+(t), \qquad r = \#\mathcal{V}_\mathfrak{b}, \qquad r \geqslant 1.$$

It is known that

$$\sigma_k(H_0) \subset \big[\mu_k^-, \mu_k^+\big], \qquad \mu_k^\pm = \mu_k^\pm(0), \quad k = 1, \ldots, \nu - r.$$

Fabila-Carrasco, J.S.; Lledó, F.; Post, O. Spectral gaps and discrete magnetic Laplacians, Linear Algebra Appl., 547 (2018), no. 15, 183–216.

We assume that

$$\mu_k^+ < \mu_{k+1}^- \quad \text{for some} \qquad k = 1, \ldots, \nu - r,$$

i.e., the interval $\mathcal{I}_k = (\mu_k^+, \mu_{k+1}^-) \neq \varnothing$ is a gap (not necessarily maximal) of H_0.

Main results

$$\mathcal{J}_k = (\mu_k^+, \mu_{k+1}^-) \neq \varnothing \quad \text{for some} \quad k = 1, \ldots, \nu - r \qquad (4)$$

is a spectral gap of the unperturbed Schrödinger operator H_0.

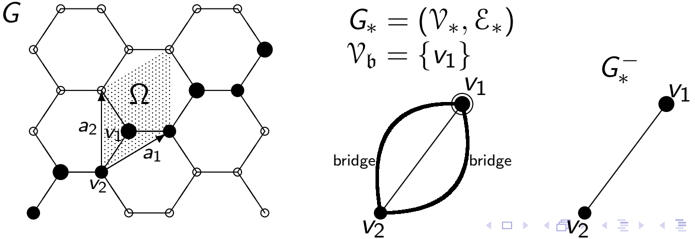
Theorem 1. Let the gap condition (4) be fulfilled, $\lambda \in \mathcal{J}_k$, and $\mathcal{V}_* \subset \operatorname{supp} Q$. Then there exist exactly k *guided band branches*

$$\sigma_{k,j}^g(t) \equiv \sigma_{k,j}^g(H_t), \qquad j = 1, \ldots, k,$$

of the operator family $H_t = H_0 + tQ$, $t > 0$, crossing the level λ. Moreover, in the gap $\mathcal{J}_k$ each of these branches satisfies

$$\sigma_{k,j}^g(t) \subset \big[\mu_j^-(t), \mu_j^+(t)\big], \qquad j = 1, \ldots, k,$$

where $\mu_j^\pm(t)$ are eigenvalues of the operators $H_t^\pm$.



Main results

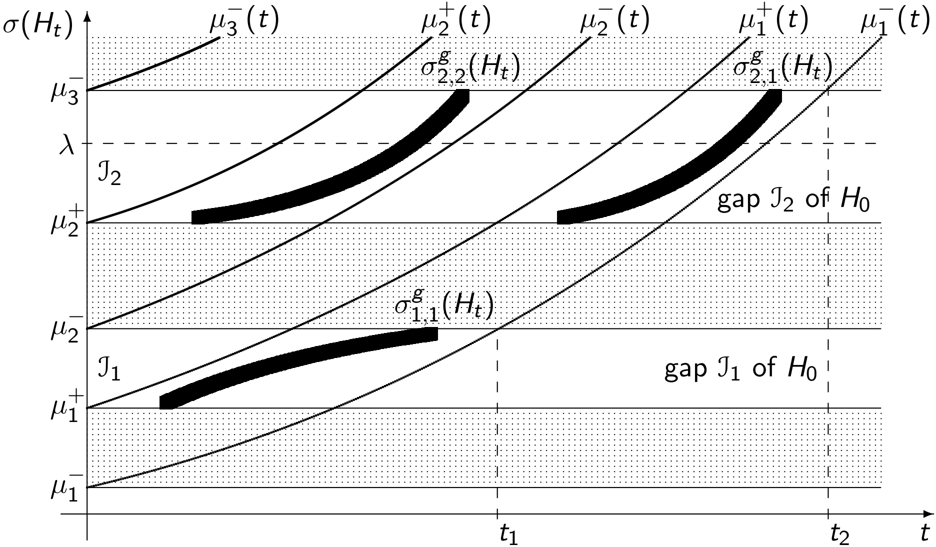


Figure: Dependence of the guided bands $\sigma^g_{k,j}(H_t)$ of $H_t = H_0 + tQ$ on $t \geqslant 0$ (the guided band branches $\sigma^g_{k,j}(t) \equiv \sigma^g_{k,j}(H_t)$) in gaps of H_0.

Main results

$$\mathcal{J}_k = (\mu_k^+, \mu_{k+1}^-) \neq \varnothing \quad \text{for some} \quad k = 1, \ldots, \nu - r, \qquad (5)$$

is a spectral gap of the unperturbed Schrödinger operator H_0; $\nu = \#\mathcal{V}_*$, $r = \#\mathcal{V}_\mathfrak{b}$.

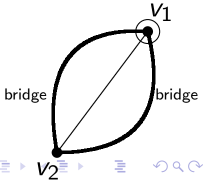
We denote by p the number of vertices in $\operatorname{supp} Q\lceil_{\mathcal{V}_*\setminus\mathcal{V}_\mathfrak{b}}$:

$$p = \#\operatorname{supp} Q\lceil_{\mathcal{V}_*\setminus\mathcal{V}_\mathfrak{b}}, \qquad 0 \leqslant p \leqslant \nu - r. \qquad (6)$$

Theorem 2. Let the gap condition (5) be fulfilled, p be defined in (6). Then the gap $\mathcal{J}_k$ contains at most $s = \min\{k, p\}$ *guided band branches* $\sigma_m^g(t) \equiv \sigma_m^g(H_t)$ of the operator family $H_t = H_0 + tQ$, $t > 0$.

$G_* = (\mathcal{V}_*, \mathcal{E}_*)$

$\mathcal{V}_\mathfrak{b} = \{v_1\}$



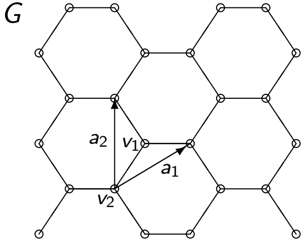
Remarks. 1) If $\operatorname{supp} Q\lceil_{\mathcal{V}_*} \subset \mathcal{V}_\mathfrak{b}$ (i.e., $p = 0$), then there are no guided band branches of H_t, $t > 0$, in $\mathcal{J}_k$, i.e., $\mathcal{J}_k$ is a gap of H_t for each $t > 0$.

2) The spectrum of H_t, of course, does not depend on the choice of the set $\mathcal{V}_b$. But the interval $\mathcal{J}_k$, in general, depends on this choice.

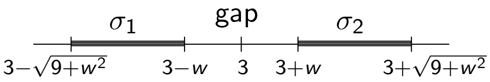
Example (the hexagonal lattice)

First we consider the unperturbed Schrödinger operator $H_0 = \Delta + W$ with a periodic potential W on the hexagonal lattice G. Without loss of generality we may assume that

$$W(v_1) = w, \qquad W(v_2) = -w, \qquad w > 0. \tag{7}$$



Spectrum of H_0



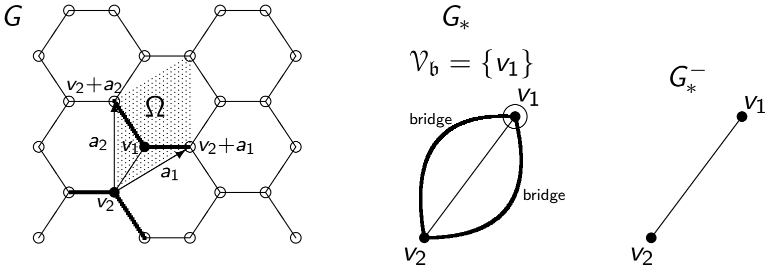
The spectrum of H_0 on G has the form

$$\sigma(H_0) = \sigma_1 \cup \sigma_2 = [\lambda_1^-, \lambda_1^+] \cup [\lambda_2^-, \lambda_2^+],$$

$$\lambda_1^- = 3 - \sqrt{9 + w^2}\,, \qquad \lambda_1^+ = 3 - w,$$

$$\lambda_2^- = 3 + w, \qquad \lambda_2^+ = 3 + \sqrt{9 + w^2}\,.$$

Example (the hexagonal lattice)



The operators $H_0^{\pm}$ have the form

$$H_0^- = \begin{pmatrix} 1+w & -1 \\ -1 & 1-w \end{pmatrix}, \qquad H_0^+ = (3-w).$$

The eigenvalues μ_1^-, μ_2^- of H_0^- and μ_1^+ of H_0^+ are given by

$$\mu_j^- = 1 + (-1)^j \sqrt{1+w^2}, \quad j = 1, 2, \qquad \mu_1^+ = 3 - w.$$

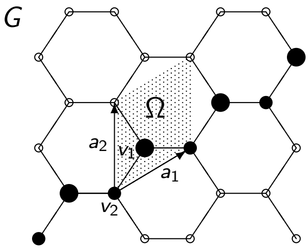
If $w > 3/4$, then $\mu_1^+ < \mu_2^-$ and the interval $\mathcal{I}_1 = (\mu_1^+, \mu_2^-) \neq \varnothing$ is a gap in the spectrum of the unperturbed operator H_0.

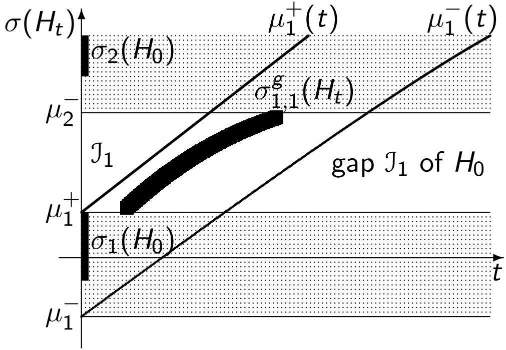
Example (the hexagonal lattice)

We consider the perturbed Schrödinger operators $H_t = H_0 + tQ$, $t > 0$, on the hexagonal lattice G with the guided potential Q satisfying the conditions

$$\operatorname{supp} Q \subset \mathbb{R} \times [0,1), \qquad Q(v + a_1) = Q(v), \qquad \forall\, v \in \mathcal{V}.$$

Case 1. $\operatorname{supp} Q\restriction_\Omega = \{v_1, v_2\}$





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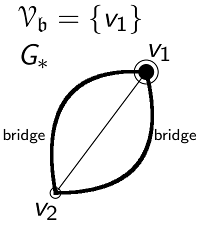
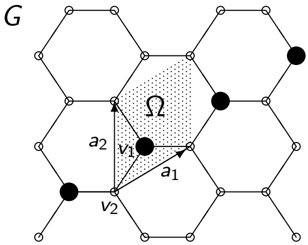
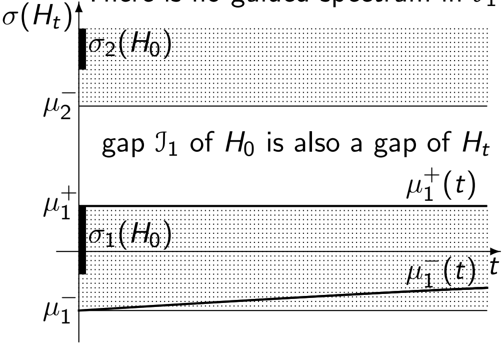
Case 2. $\operatorname{supp} Q\restriction_\Omega = \{v_1\} = \mathcal{V}_\mathfrak{b}$





There is no guided spectrum in $\mathfrak{J}_1$



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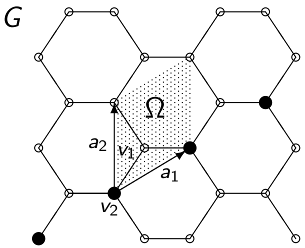
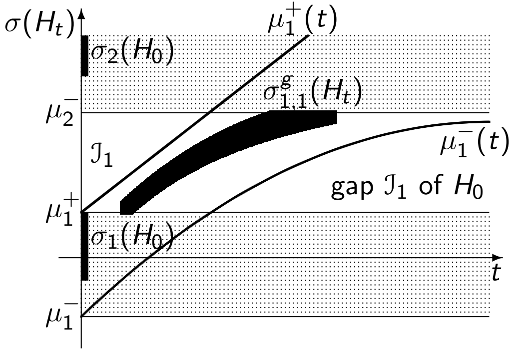
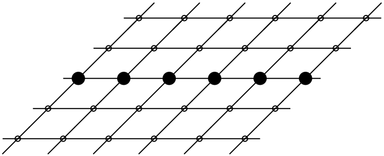
Case 3. $\operatorname{supp} Q\restriction_\Omega = \{v_2\}$





References



1) The square lattice with an infinite line defect was considered in

Colquitt, D.J.; Nieves, M.J., Jones, I.S.; Movchan, A.B., and Movchan, N.V., Localization for a line defect in an infinite square lattice, Proc. R. Soc. A, 469 (2013), 20120579.

Osharovich, G.G.; Ayzenberg-Stepanenko, M.V. Wave localization in stratified square-cell lattices: The antiplane problem, J. Sound Vib., 331 (2012), 1378–1397.

In this case the dispersion relations for defect modes generated by a line defect can be computed in explicit form.

2) The procedure for calculating the spectrum of discrete periodic operators perturbed by periodic operators of smaller dimensions was given in

Kutsenko, A. Algebra of multidimensional periodic operators with defects. J. Math. Anal. Appl. 428.1 (2015), 217–226.

The algorithm is based on algebraic operations on the finite matrix-valued functions and integration.

Thank you for attention!