Optimal potentials on quantum graphs with δ-couplings

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joint work with Pavel Kurasov

Differential Operators on Graphs and Waveguides
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Overview

Our work concerns the study of the supremum of the first eigenvalue of the Schrödinger operator $-\frac{d^2}{dx^2} + q(x)$ with $q \in L_1$ with fixed total mass Q and Robin conditions on metric graphs.

- The problem was originally formally posed by A. G. Ramm in '82 for the Dirichlet case on the interval.
- The case of Dirichlet boundary condition on the interval has been studied by G.Talenti '84, E.M.Harrell '84, M.Essèn '87, Egnell '87, V. A. Vinokurov and V. A. Sadovnichii '03 and S.S.Ezhak '07.
- The case with fixed Robin conditions on the interval were considered by E.S. Karulina and A.A. Vladimirov '13.

The operator

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- We consider delta vertex conditions (δ-**v.c.**) at the vertices $v \in V$

$$\begin{cases} \psi \text{ is continuous in } v, \\ \sum_{x_j \in v} \partial\psi(x_j) = h(v)\psi(v), \end{cases}$$

where $\partial\psi(x_j)$ denotes the **normal derivative** of ψ.

Hence $L_q^h(\Gamma)$ acts on the following domain

$$\mathcal{D}\left(L_q^h(\Gamma)\right) := \left\{ u \in \left(\bigoplus_{n=1}^{N} \mathcal{W}_2^1(e_n)\right) \cap \mathcal{C}(\Gamma) : \ -\frac{d^2}{dx^2}u|_{e_n} + qu|_{e_n} \in L_2(e_n), \forall n; \right.$$
$$\left. \sum_{x_i \in v} \partial u(x_i) = h(v)u(v) \, \forall v \in V(\Gamma) \right\},$$

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Spectrum of Compact Finite Quantum Graphs

Proposition (The spectrum is discrete)

The spectrum of the Schrödinger operator $L_q^h(\Gamma)$ with L_1-potential q and with real δ-vertex conditions h is discrete.
$\sigma(L_q^h(\Gamma)) = \{\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots\}$.

If the graph is connected then the first eigenvalue is simple:
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$$\lambda \leq \lambda_1 \left(L_q^h(\Gamma)\right) \leq \Lambda \tag{1}$$

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- Eigenvalues inequalities for the Laplacian on metric graphs ($q \equiv 0$).
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- Lower bound of the Ground state of Quantum Graphs ($q \neq 0$)

Theorem (Karreskog, Kurasov, Kupersmidt '15)

Let $L_q^h(\Gamma)$ be a Schrödinger operator on a finite compact metric graph Γ with the total negative strength $I^- := \int q^- + \sum h^-$, and the total positive strengths $I^+ := \int q^+ + \sum h^+$. Then

$$\lambda_1(L_0^I([0,\mathcal{L}])) \leq \lambda_1(L_q^h(\Gamma))$$

where on the interval $[0,\mathcal{L}]$ the following δ-vertex conditions

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The optimisation problem.

Given a graph Γ we are interested in

$$\Lambda(\Gamma, Q, H) := \sup_{\substack{(q,h) \\ \text{Under ass. 1,2,3.}}} \lambda_1 \left(L_q^h(\Gamma) \right) \tag{2}$$

– **the optimal upper bound** under the following assumptions:

Assumption

1. $\int_\Gamma q(x)\,dx = Q$, *(the total strength of the potential be fixed),*
2. $\sum_{v \in V} h(v) = H$, *(the total strength of the singular interaction be fixed),*
3. *the potential is sign-definite:* $\begin{cases} q(x) \geq 0 & \text{if } Q \geq 0, \\ q(x) \leq 0 & \text{if } Q \leq 0, \end{cases}$ $\forall x \in \Gamma$.

Main result

Our main results can be formulated as follows:

- The optimisation problem is independent of the topology of the graph, hence it is enough to study flower graphs.
- If $Q \cdot H \geq 0$, then the **optimal configuration** (q^*, h^*) exists and is unique. It is described by explicit formulas.
- If $Q \cdot H < 0$, then the optimal configuration does not exist, but the value of the optimal ground state energy can either be given explicitly by showing an **optimising sequence** (q_n, h_n) or as an eigenvalue of the Laplacian on a flower graph with delta interactions.

Perron-Frobenius Theorem

Proposition (Perron-Frobenius theorem for quantum graphs)

The ground state may be chosen strictly positive $\psi_1 > 0$. Moreover, the corresponding eigenvalue is simple (Γ is connected).

Corollary

Let ψ be a real nonnegative eigenfunction of $L_q^h(\Gamma)$, then $\psi = \psi_1$, i.e. it is the ground state eigenfunction.

Idea of the proof: Let $\psi_1 > 0$ be the GS with $\lambda_1 \neq \lambda$ and use the orthogonality of the eigenfunctions to reach a contradiction.

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Associated quadratic form and Rayleigh quotient

We denote by $Q_{L_q^h(\Gamma)}$ the quadratic form associated with $L_q^h(\Gamma)$, given by

$$Q_{L_q^h(\Gamma)}(u) = \|u'\|^2 + \int_\Gamma q(x)|u(x)|^2\,dx + \sum_{v\in V(\Gamma)} h(v)|u(v)|^2, \tag{3}$$

on the domain

$$\mathcal{D}\left(Q_{L_q^h(\Gamma)}\right) = \left(\bigoplus_{n=1}^{N} \mathcal{W}_2^1(e_n)\right) \cap \mathcal{C}(\Gamma) \supsetneq \mathcal{D}\left(L_q^h(\Gamma)\right).$$

Proposition (Rayleigh quotient)

$$\lambda_1\left(L_q^h(\Gamma)\right) = \min_{u\in\mathcal{D}\left(Q_{L_q^h(\Gamma)}\right)} \frac{Q_{L_q^h(\Gamma)}(u)}{\|u\|^2}.$$

The minimiser coincides with the ground state $\psi_1^{L_q^h(\Gamma)}$ *(* $=\psi_1^q$ *when* h *is fixed)*

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Sufficient condition for optimality

Lemma (Optimality criterion 1)

Let the delta couplings (h*) be fixed. Assume that* $q^* \geq 0$ *such that*

$$\textit{supp}\, q^* \subseteq M_{q^*} = \left\{ x \in \Gamma : \psi_1^{q^*}(x) = \max_{y \in \Gamma} \psi_1^{q^*}(y) \right\}, \tag{4}$$

then

- *the potential* q^* *is optimal in the sense, that*

$$\lambda_1(L^h_{q^*}(\Gamma)) \geq \lambda_1(L^h_q(\Gamma)) \tag{5}$$

  holds $\forall q \geq 0$ *with the same total strength* $\int_\Gamma q^*(x)dx = \int_\Gamma q(x)dx$.
- *The optimal potential* q^* *is unique.*

Sufficient condition for optimality

Lemma (Criterion for optimality)

If $\operatorname{supp} q^* \subseteq M_{q^*}$, *then*

$$\lambda_1\left(L^h_q(\Gamma)\right) \leq \lambda_1\left(L^h_{q^*}(\Gamma)\right) \quad \forall q \in \left\{q \in L_1(\Gamma) : \int_\Gamma q(x)\,dx = Q\right\}.$$

Proof.

$$\lambda_1(L^h_{q^*}) = Q^h_{q^*}(\Gamma, \psi_1^{q^*}) \tag{6}$$

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$$\lambda_1(L_{q^*}^h) = Q_{q^*}^h(\Gamma, \psi_1^{q^*})$$

$$= \|\psi_1^{*\prime}\|_2^2 + \int_\Gamma q^*|\psi_1^*|^2\,dx + \sum_{v\in V} h(v)|\psi_1^*|^2(v) \tag{6}$$

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$$\lambda_1(L^h_{q^*}) = Q^h_{q^*}(\Gamma, \psi_1^{q^*}) \geq Q^h_q(\Gamma, \psi_1^{q^*})$$

$$= \|\psi_1^{*\prime}\|_2^2 + Q\sup_\Gamma|\psi_1^*|^2 + \sum_{v\in V} h(v)|\psi_1^*|^2(v) \tag{6}$$

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Proof.

$$\lambda_1(L^h_{q^*}) = Q^h_{q^*}(\Gamma, \psi_1^{q^*}) \geq Q^h_q(\Gamma, \psi_1^{q^*}) \geq \min_{\|\psi\|=1} Q^h_q(\Gamma, \psi) = \lambda_1(L^h_q).$$

$$\geq \|\psi_1^{*\prime}\|_2^2 + \int_\Gamma q|\psi_1^*|^2\,dx + \sum_{v\in V} h(v)|\psi_1^*|^2(v) \tag{6}$$

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$$\geq \min_{\|\psi\|=1}\left(\|\psi\|_2^2 + \int_\Gamma q|\psi|^2\,dx + \sum_{v\in V} h(v)|\psi|^2(v)\right) \tag{6}$$

□

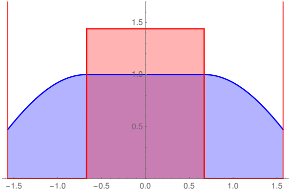
Optimal configuration on the loop graph

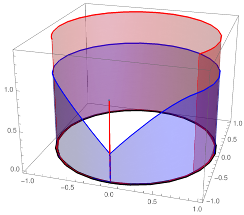
Theorem (Kurasov,S.)

On the loop graph $L = [-\frac{\ell}{2}, +\frac{\ell}{2}]$ *with* $H, Q \geq 0$ *there exists an optimal configuration* (q^*, h^*)

$$q^*(x) = \begin{cases} 0 \\ k^2 \end{cases} \qquad \psi^*(x) = \begin{cases} \cos\left(k\left(|x| - \frac{\ell}{2} + \alpha\right)\right) & |x| \geq \frac{\ell}{2} - \alpha \\ 1 & |x| < \frac{\ell}{2} - \alpha \end{cases}$$

where α *and* k *are the smallest positive solutions of* $\begin{cases} k^2(\ell - 2\alpha) = Q \\ 2k\tan(k\alpha) = H \end{cases}$





Optimal configuration on flower graphs

- $Q_n > 0$ then

$$q_{\alpha,k}(x) = \begin{cases} 0, \\ k^2, \end{cases} \qquad \psi_{\alpha,k}(x) = \begin{cases} \cos\left(k\left(|x| - \left(\alpha - \frac{\ell_n}{2}\right)\right)\right), & |x| \geq \alpha - \frac{\ell_n}{2}; \\ 1, & |x| \leq \alpha - \frac{\ell_n}{2}, \end{cases} \tag{7}$$

- $Q_n = 0$ then

$$q_{\alpha,k}(x) \equiv 0, \qquad \psi_{\alpha,k}(x) = \frac{\cos k\alpha}{\cos k\ell_n/2} \cos kx, \tag{8}$$

- $H_n = 2k \tan k\ell_n/2.$

where α is a common real parameter.
We calculate the corresponding total potential and total singular interaction and put them equal to Q and H respectively and get the following system

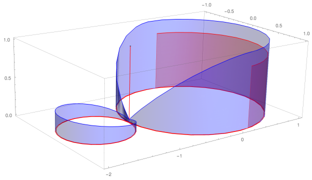
$$\begin{cases} k^2 \sum\limits_{\ell_n \geq 2\alpha} (\ell_n - 2\alpha) = Q, \\ \sum\limits_{\ell_n \geq 2\alpha} 2k \tan k\alpha + \sum\limits_{\ell_n < 2\alpha} 2k \tan k\ell_n/2 = H. \end{cases} \tag{9}$$

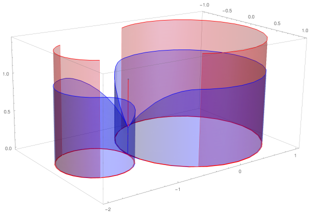
Optimal configuration on flower graphs

Theorem

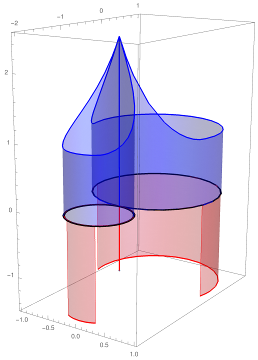
Any flower graph F, for any given $H, Q \geq 0$ total strength of the interactions, admits an optimal potential q with corresponding optimal ground state $\psi > 0$ such that

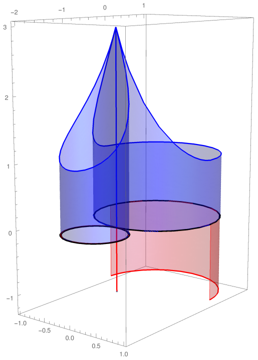
1. $\lambda_1 \left(L_q^H(F) \right) = \Lambda(F, H, Q)$,
2. *the pair q, ψ satisfies the optimality criterion:* $\operatorname{supp} q \subseteq M_q$,
3. *the restriction of q, ψ on each of the petals are symmetric,*
4. *the ground state ψ has all its oriented derivatives nonnegative.* $\partial\psi(x_j) \geq 0$





Negative case ($Q < 0, H < 0$)





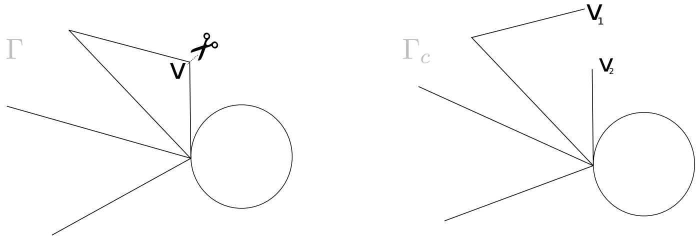
Surgery on Quantum Graphs

Definition (cutting with respect to ψ without disconnecting the graph)

Let $\psi \in \mathcal{D}\left(L_q^h(\Gamma)\right)$ and let $v \in V(\Gamma), v_1, v_2 \in V(\Gamma_c), v = v_1 \cup v_2$.
We define the restriction of the δ-vertex conditions h to h_c on Γ_c in the following way

$$h_c(w) := \begin{cases} h(w) & \text{if } w \in V(\Gamma_c) \setminus \{v_1, v_2\}, \\ \frac{1}{\psi(v)} \sum_{x_j \in v_i} \partial\psi(x_j) & \text{if } w = v_1, v_2. \end{cases} \tag{10}$$

therefore we say that the graph Γ **is cut in** v **into** Γ_c **with respect to** ψ.



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Definition (glueing two vertices of a graph)

Let $v_i, v_j \in V(\Gamma)$.
We define the new δ-vertex conditions h_g on Γ_g in the following way

$$h_g(w) := \begin{cases} h(w) & \text{if } w \in V(\Gamma_c) \setminus \{v_1, v_2\}, \\ h(v_1) + h(v_2) & \text{if } w = v_1 \cup v_2. \end{cases} \tag{11}$$

therefore we say that the graph Γ_g **is obtained by glueing** $v_i, v_j \in V(\Gamma)$ **into** v.

Cutting/glueing a graph vs the ground energy

Lemma (cutting)

$$\lambda_1 \left(L_q^h(\Gamma) \right) = \lambda_1 \left(L_q^{h_c}(\Gamma_c) \right)$$

Idea of the proof.

Apply the corollary of Perron-Frobenius Theorem on Γ_c and ψ. □

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Lemma (glueing)

$$\Lambda(\Gamma,H,Q)\le\Lambda(\Gamma_g,H,Q)$$

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Notice that by the definition of glueing $\mathcal{C}(\Gamma_g)\subsetneq\mathcal{C}(\Gamma)$;
Hence by glueing the domain shrinks
$\mathcal{D}_g:=\mathcal{D}\left(Q_{L_q^{h_g}(\Gamma_g)}\right)\subsetneq\mathcal{D}:=\mathcal{D}\left(Q_{L_q^h(\Gamma)}\right)$ and
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Cutting/glueing a graph vs the ground energy

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$$\lambda_1\left(L_q^h(\Gamma)\right)=\lambda_1\left(L_q^{h_c}(\Gamma_c)\right)$$

Idea of the proof.

Apply the corollary of Perron-Frobenius Theorem on Γ_c and ψ. □

Lemma (glueing)

$$\Lambda\left(\Gamma,H,Q\right)\leq\Lambda\left(\Gamma_g,H,Q\right)$$

Idea of the proof.

Notice that by the definition of glueing $\mathcal{C}\left(\Gamma_g\right)\subsetneq\mathcal{C}\left(\Gamma\right)$;
Hence by glueing the domain shrinks
$\mathcal{D}_g:=\mathcal{D}\left(Q_{L_q^{h_g}(\Gamma_g)}\right)\subsetneq\mathcal{D}:=\mathcal{D}\left(Q_{L_q^h(\Gamma)}\right)$ and
$Q_q^{h_g}(\Gamma_g,\psi)=Q_q^h(\Gamma,\psi),\ \forall\psi\in\mathcal{D}_g$.The inequality follows from comparing the minimum of the Rayleigh quotient over the domains $\mathcal{D}_g,\mathcal{D}$ □

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Optimality for arbitrary graphs

Theorem (Kurasov,S.)

Let F_Γ be the flower graph associated to a given graph Γ.

$$\Lambda(\Gamma, Q, H) = \Lambda(F_\Gamma, Q, H), \quad \forall Q, H \in \mathbb{R}. \tag{12}$$

In particular, if q^* is the optimal potential on F_Γ then (q^*, h') is the optimal configuration on Γ, where h' is obtained by cutting F_Γ into Γ along the flower ground state. Moreover the same function is a ground state for $L_{q^*}^{h'}(\Gamma)$.

$$\begin{array}{ccc} \Lambda(\Gamma, Q, H) & \leq & \Lambda(F_\Gamma, Q, H) \\ \vee\mid & & \parallel \\ \lambda_1(L_{q^*}^{h'}(\Gamma)) & = & \lambda_1\left(L_{q^*}^{H}(F_\Gamma)\right) \end{array} \qquad \begin{array}{ccc} \Lambda(\Gamma, Q, H) & \leq & \Lambda(F_\Gamma, Q, H) \\ \vee\mid & & \uparrow n \to \infty \\ \lambda_1(L_{q_n}^{h'_n}(\Gamma)) & = & \lambda_1\left(L_{q_n}^{H}(F_\Gamma)\right) \end{array}$$

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- Moreover, if the potential is **dominant** over the singular interaction then $\Lambda(\Gamma, H, Q)$ can be computed just from
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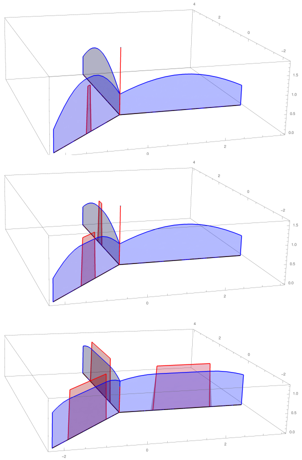

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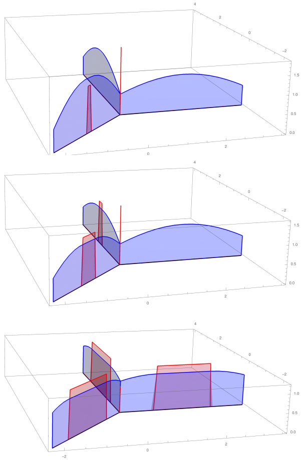

Two non-optimal cases, $Q \cdot H < 0$

Lemma $(0 < |H| < |Q|)$

If $0 < -H \leq Q$ *or* $0 < H \leq -Q$ *then there is no optimal configuration and*

$$\Lambda(\Gamma, Q, H) = \frac{Q + H}{\mathcal{L}(\Gamma)}.$$

Lemma $(0 < |Q| < |H|)$

If $0 < -Q \leq H$ *or* $0 < Q \leq -H$ *then no optimal configuration exists and*

$$\Lambda(\Gamma, Q, H) = \lambda_1 \left(L_0^{Q+H}(F_\Gamma) \right).$$

Thanks for your attention!