Approximation of Laplacians on the Sierpinski Gasket: norm-resolvent and spectral convergence

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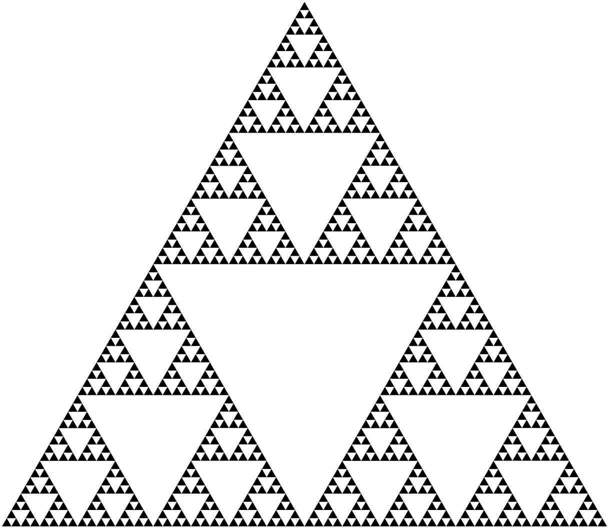

5th level iteration of the Sierpsinski Gasket

The Sierpinski Gasket

Definition (Sierpinski Gasket)

Let p_1, p_2 and p_3 be the vertices of an equilateral triangle in $\mathbb{R}^2$ and

$$F_j\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad F_j(x) = (x - p_j)/2 + p_j \qquad (j = 1, 2, 3).$$

Then we call the unique non-empty compact $\mathcal{K} \subset \mathbb{R}^2$ that satisfies

$$\mathcal{K} = F_1(\mathcal{K}) \cup F_2(\mathcal{K}) \cup F_3(\mathcal{K})$$

the *Sierpinski Gasket*. Moreover, we call $V_0 := \{p_1, p_2, p_3\}$ the *boundary of SG*.

Let $W_m := \{1, 2, 3\}^m$. Then there is a natural cell structure on SG given by

$$W_m \ni w \mapsto F_w(\mathcal{K}) := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}(\mathcal{K}).$$

We call $F_w(\mathcal{K})$ an *m-cell of $\mathcal{K}$*.

Approximating sequence of finite graphs for SG

Definition

We let $G_0 := (V_0, E_0)$ be the complete graph and for $m \in \mathbb{N}$ we define a sequence of finite discrete graphs $G_m = (V_m, E_m)$ by

$$V_m := \bigcup_{w \in W_m} F_w(V_0), \quad E_m := \left\{ \{x, y\} \subset V_m \,\middle|\, x \sim_m y \right\},$$

where $x \sim_m y \iff x \neq y$ and $\exists w \in W_m$ such that $x, y \in F_w(K)$.

Note that $V_m \subset V_{m+1}$ for each $m \in \mathbb{N}_0$ and

$$V_\star := \bigcup_{m \in \mathbb{N}_0} V_m \subset \mathcal{K} \text{ dense.}$$

Note also that SG is connected and

$$F_w(\mathcal{K}) \cap F_{w'}(\mathcal{K}) \subset F_w(V_0) \cap F_{w'}(V_0) \qquad \forall m \in \mathbb{N}, w \neq w' \in W_m.$$

Energy forms on the approximating graphs

Definition

On each graph $G_m = (V_m, E_m)$ we define an energy form by

$$\mathcal{E}_m(f) = \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} \left|f(x) - f(y)\right|^2$$

for $f\colon V_m \to \mathbb{C}$.

The constant $(5/3)^m$ is chosen such that the minimisation problem

$$\mathcal{E}_m(\varrho) = \min\left\{ \mathcal{E}_{m+1}(f) \,\middle|\, f\colon V_{m+1} \to \mathbb{C}, f\restriction_{V_m} = \varrho \right\}$$

has a unique solution for each $\varrho\colon V_m \to \mathbb{C}$.

Energy form on SG

Let $u\colon V_\star \to \mathbb{C}$. As $u\restriction_{V_m}$ is any extension of $u\restriction_{V_{m-1}}$ and we have

$$\mathcal{E}_{m-1}(u\restriction_{V_{m-1}}) \le \mathcal{E}_m(u\restriction_{V_m})$$

and hence the following limit exists in $[0,\infty]$:

$$\mathcal{E}_\infty(u) := \lim_{m\to\infty} \mathcal{E}_m(u\restriction_{V_m}).$$

Theorem ([Ki01] Energy form on SG)

There exists an energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on SG related to the sequence $\{(G_m, \mathcal{E}_m)\}_{m\in\mathbb{N}_0}$ given by $\mathcal{E} = \mathcal{E}_\infty$ with domain

$$\operatorname{dom}\mathcal{E} := \left\{\, u \in \mathsf{C}(\mathcal{K}) \,\middle|\, \mathcal{E}(u) := \lim_{m\to\infty} \mathcal{E}_m(u\restriction_{V_m}) < \infty \,\right\}$$

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There exists an energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *on* SG *related to the sequence* $\{(G_m,\mathcal{E}_m)\}_{m\in\mathbb{N}_0}$ *given by* $\mathcal{E}=\mathcal{E}_\infty$ *with domain*

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Harmonic functions

The compatibility of the sequence $\{\mathcal{E}_m\}_{m\in\mathbb{N}_0}$ implies:

Theorem ([Ki01] m-harmonic functions on SG)

For any boundary value $\varrho\colon V_m \to \mathbb{C}$ *there exists a unique function* $h \in \operatorname{dom}\mathcal{E}$ *such that* $h\restriction_{V_m} = \varrho$ *and*

$$\mathcal{E}_m(\varrho) = \mathcal{E}(h) = \min\left\{\, \mathcal{E}(u) \,\middle|\, u \in \operatorname{dom}\mathcal{E}, u\restriction_{V_m} = \varrho \,\right\}.$$

The function h is called *m-harmonic function with boundary values ϱ*. In the special case where $\varrho = \mathbb{1}_{\{x\}}$ for $x \in V_m$, we denote the corresponding m-harmonic function by $\psi_{x,m}$.

Specifying the Hilbert spaces

Let μ be the *(homogeneous) self-similar (probability) measure* on SG, i.e. for all Borel sets $A \subset \mathcal{K}$,

$$\mu(A) = \frac{1}{3}\big(\mu(F_1^{-1}(A)) + \mu(F_2^{-1}(A)) + \mu(F_3^{-1}(A))\big).$$

Hence every m-cell has measure $\mu(\mathcal{K}_w) = 1/3^m$.

Then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a densely defined, closed quadratic form in $\mathsf{L}_2(\mathcal{K}, \mu)$ and we denote the corresponding non-negative and self-adjoint operator by Δ.

On $G_m = (V_m, E_m)$ we define a probability measure by

$$\mu_m(x) := \int_{\mathcal{K}} \psi_{x,m} \, \mathrm{d}\mu = \begin{cases} 1/3^{m+1} & x \in V_0 \\ 2/3^{m+1} & x \in V_m \setminus V_0. \end{cases}$$

Then our Hilbert space structure is $\mathscr{H}_m = \ell_2(V_m, \mu_m)$ with norm

$$\|f\|^2_{\ell_2(V_m,\mu_m)} = \sum_{x \in V_m} \mu_m(x)|f(x)|^2.$$

It is easy to see that $\Delta_m \geq 0$ acts as

$$\Delta_m f(y) = \frac{1}{\mu_m(y)} \sum_{x \underset{m}{\sim} y} \left(\frac{5}{3}\right)^m \big(f(y) - f(x)\big) = \frac{3}{2} 5^m \sum_{x \underset{m}{\sim} y} \big(f(y) - f(x)\big).$$

Problem: We have energy forms $\mathcal{E}_m$ in $\ell_2(V_m, \mu_m)$ and an energy form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ in $\mathsf{L}_2(\mathcal{K}, \mu)$ and the spaces are all different. How can we give any sense to the following expression?

$$\|(\Delta_m + 1)^{-1} - (\Delta + 1)^{-1}\| \to 0$$

Generalised norm resolvent convergence

Let $(\mathcal{E}_m, \mathscr{H}_m^1)$ resp. $(\mathcal{E}, \mathscr{H}^1)$ be energy forms in the separable Hilbert spaces $\mathscr{H}_m$ resp. $\mathscr{H}$.

Definition ([P12] Quasi-unitary equivalence)

Let $\delta_m \geq 0$. Then $\mathcal{E}_m$ and $\mathcal{E}$ are called *δ_m-quasi-unitary equivalent* if there exist $J_m \colon \mathscr{H}_m \to \mathscr{H}$, $J_m^1 \colon \operatorname{dom} \mathcal{E}_m \to \operatorname{dom} \mathcal{E}$ and $J_m'^1 \colon \operatorname{dom} \mathcal{E} \to \operatorname{dom} \mathcal{E}_m$ such that $\|J_m f\|_{\mathscr{H}} \leq (1 + \delta_m)\|f\|_{\mathscr{H}}$ and

$$\|f - J_m^\star J_m f\|_{\mathscr{H}_m} \leq \delta_m \|f\|_{\mathcal{E}_m} \qquad \|u - J_m J_m^\star u\|_{\mathscr{H}} \leq \delta_m \|u\|_{\mathcal{E}}$$

$$\|J_m f - J_m^1 f\|_{\mathscr{H}} \leq \delta_m \|f\|_{\mathcal{E}_m} \qquad \|J_m^\star u - J_m'^1 u\|_{\mathscr{H}_m} \leq \delta_m \|u\|_{\mathcal{E}}$$

$$|\mathcal{E}(J_m f, u) - \mathcal{E}_m(f, J_m'^1 u)| \leq \delta_m \|f\|_{\mathcal{E}_m} \|u\|_{\mathcal{E}}$$

where $\|u\|_{\mathcal{E}}^2 := \|u\|_{\mathscr{H}}^2 + \mathcal{E}(u)$.

Theorem

If $\mathcal{E}_m$ and $\mathcal{E}$ are δ_m-quasi-unitary equivalent then

$$\|J_m(\Delta_m + 1)^{-1} - (\Delta + 1)^{-1} J_m\| \leq 4\delta_m.$$

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$$\|f - J_m^\star J_m f\|_{\mathscr{H}_m} \leq \delta_m \|f\|_{\mathcal{E}_m} \qquad \|u - J_m J_m^\star u\|_{\mathscr{H}} \leq \delta_m \|u\|_{\mathcal{E}}$$

$$\|J_m f - J_m^1 f\|_{\mathscr{H}} \leq \delta_m \|f\|_{\mathcal{E}_m} \qquad \|J_m^\star u - J_m'^1 u\|_{\mathscr{H}_m} \leq \delta_m \|u\|_{\mathcal{E}}$$

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Consequences of quasi-unitary equivalence

Theorem ([P12])

Assume that $\mathcal{E}$ and $\mathcal{E}_m$ are δ_m-quasi-unitarily equivalent and that U is an open subset such that ∂U is locally Lipschitz and $\partial U \cap (\sigma(\Delta_m) \cup \sigma(\Delta)) = \emptyset$. Then

$$\|\eta(\Delta) J_m - J_m \eta(\Delta_m)\| \leq C_\eta \delta_m$$

for any holomorphic $\eta \colon U \to \mathbb{C}$, where the constants C_η only depend on η and U.

For example choose $\eta(\lambda) = \mathrm{e}^{-t\lambda}$ then the theorem is about the norm convergence of the approximating heat operators on (G_m, μ_m) to the one on the SG.

Consequences of quasi-unitary equivalence

If $\eta = \mathbb{1}_I$ ($\partial I \cap \sigma(\Delta) = \emptyset$), then the above theorem states the convergence of the spectral projectors and we conclude:

Corollary ([P12])

Let $\lambda_k(\Delta_m)$ resp. $\lambda_k(\Delta)$ be the k-th eigenvalue of Δ_m resp. Δ. Then

$$|\lambda_k(\Delta_m) - \lambda_k(\Delta)| \leq C_k \delta_m$$

for all $m \in \mathbb{N}$ such that $\dim \mathscr{H}_m \geq k$ and where C_k only depends on $\lambda_k(\Delta)$.

Since the spectrum of Δ is purely discrete we can approximate an eigenfunction also in energy norm: For $\lambda \in \sigma(\Delta)$ with normalised eigenfunction Φ there is a sequence $(\Phi_m)_m$ of normalised function (linear combinations of eigenfunctions with eigenvalues close to Δ) and $C_\lambda > 0$ (only depending in λ) such that

$$\|J_m \Phi_m - \Phi\|_{\operatorname{dom} \mathcal{E}} \leq C_\lambda \delta_m.$$

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$$\|J_m \Phi_m - \Phi\|_{\mathrm{dom}\,\mathcal{E}} \leq C_\lambda \delta_m.$$

Main results

In our setting on the SG, this means:

1. $\mathscr{H}_m := \ell_2(V_m, \mu_m)$ where $\mu_m(x) := \int \psi_{x,m} \, \mathrm{d}\mu$ and
$$\mathcal{E}_m(f) := \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} \left| f(x) - f(y) \right|^2$$
2. $\mathscr{H} := \mathsf{L}_2(\mathcal{K}, \mu)$ with energy form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ defined by
$$\mathcal{E}(u) := \lim_{m \to \infty} \mathcal{E}_m(u{\restriction}_{V_m})$$
for each $u \in \{\, u \in \mathsf{C}(\mathcal{K}) \,|\, \mathcal{E}(u) := \lim_{m \to \infty} \mathcal{E}_m(u{\restriction}_{V_m}) < \infty \,\}$

Main results

Theorem ([PS18a])

$\mathcal{E}_m$ and $\mathcal{E}$ are δ_m-quasi-unitarily equivalent with

$$\delta_m = \frac{(1+\sqrt{3})\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{5^{m/2}}.$$

Flavour of the proof: We define $J := J_m \colon \mathscr{H}_m \to \mathscr{H}$ by

$$Jf = \sum_{x \in V_m} f(x)\psi_{x,m} \quad \text{then} \quad J^\star u(y) = \frac{1}{\mu_m(y)} \langle u, \psi_{y,m} \rangle_{\mathscr{H}}$$

and let $J^1 \colon \mathscr{H}_m^1 \to \mathscr{H}^1$ and $J'^1 \colon \mathscr{H}^1 \to \mathscr{H}_m^1$

$$J^1 = J\restriction_{\mathscr{H}_m^1} \quad \text{and} \quad J'^1 u(y) = u(y).$$

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Main results

Then we have

$$f(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} f(y) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}$$

and

$$J^\star J f(y) = \sum_{x \in V_m} f(x) J^\star \psi_{x,m}(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} f(x) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}$$

Hence

$$f(y) - J^\star J f(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}} \big(f(y) - f(x) \big)$$

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Hence

$$f(y) - J^\star Jf(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}} \big(f(y) - f(x) \big)$$

And then we can estimate in norm:

$$\|f - J^\star Jf\|^2_{\mathscr{H}_m} = \sum_{y \in V_m} \frac{1}{\mu_m(y)} \Big| \sum_{x \in V_m} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}} \big(f(y) - f(x)\big) \Big|^2$$

$$\leq \sum_{y \in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x \in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}^2}{(5/3)^m} \Big) \cdot \sum_{x \underset{m}{\sim} y} \Big(\frac{5}{3}\Big)^m \big| f(x) - f(y) \big|^2$$

$$\leq \underbrace{\sup_{y \in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x \in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}^2}{(5/3)^m} \Big)}_{\sim \frac{1}{5^m}} \cdot \mathcal{E}_m(f)$$

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$$\leq \sum_{y\in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x\in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}^{\ 2}}{(5/3)^m} \Big)$$

$$\cdot \sum_{x \underset{m}{\sim} y} \Big(\frac{5}{3}\Big)^m \big| f(x) - f(y) \big|^2$$

$$\leq \underbrace{\sup_{y\in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x\in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathscr{H}}^{\ 2}}{(5/3)^m} \Big)}_{\sim \frac{1}{5^m}} \cdot \mathcal{E}_m(f)$$

Main results

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$$
\begin{aligned}
\|f - J^\star Jf\|^2_{\mathscr{H}_m} &= \sum_{y\in V_m} \frac{1}{\mu_m(y)} \Big| \sum_{x\in V_m} \langle \psi_{x,m}, \psi_{y,m}\rangle_{\mathscr{H}} \big(f(y) - f(x)\big)\Big|^2 \\
&\leq \sum_{y\in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x\in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m}\rangle_{\mathscr{H}}^2}{(5/3)^m} \Big) \\
&\qquad \cdot \sum_{x \underset{m}{\sim} y} \left(\frac{5}{3}\right)^m \big| f(x) - f(y)\big|^2 \\
&\leq \underbrace{\sup_{y\in V_m} \frac{1}{\mu_m(y)} \Big(\sum_{x\in V_m} \frac{\langle \psi_{x,m}, \psi_{y,m}\rangle_{\mathscr{H}}^2}{(5/3)^m} \Big)}_{\sim \frac{1}{5^m}} \cdot \mathcal{E}_m(f)
\end{aligned}
$$

Main results: Metric graph

A *metric graph* is a discrete graph G together with an edge length function $\ell\colon E \to (0,\infty)$.

$$M = \biguplus_{e\in E} M_e/\omega, \qquad \text{where} \quad M_e = [0,\ell_e].$$

(i) A distance we choose the shortest path

(ii) A measure ν is given by the sum of the Lebesgue measures on the edges

(iii) $\mathscr{H} = \mathsf{L}_2(M,\nu)$ with norm

$$\|u\|^2_{\mathsf{L}_2(M,\nu)} = \sum_{e\in E}\int_0^{\ell_e} |u_e(x)|^2 \,\mathrm{d}x_e.$$

and energy form $(\mathcal{E}_M, \operatorname{dom}\mathcal{E}_M)$, $\operatorname{dom}\mathcal{E}_M = \mathsf{H}^1(M)$

$$\mathcal{E}_M(u) = \|u'\|^2_{\mathsf{L}_2(M,\nu)} = \sum_{e\in E}\int_0^{\ell_e} |u_e'(x_e)|^2 \,\mathrm{d}x_e$$

Main results: Metric graph

Let $\mathcal{K}$ be as before with self-similar measure μ and approximating sequence $G_m = (V_m, E_m)$. We choose

1. $M_m = (G_m, \ell_m)$, with length function $\ell_m(e) = 2^{-m}$
2. with energy form $(\tau_m \mathcal{E}_{M_m}, \operatorname{dom} \mathcal{E}_{M_m})$

$$\tau_m \mathcal{E}_{M_m}(u) = 3 \cdot \left(\frac{5}{4}\right)^m \|u'\|^2_{\mathsf{L}_2(M,\nu)}$$

3. $J_m f = c_m \sum_{x \in V_m} f(x) \widetilde{\psi}_{x,m}$ where $c_m^2 = (1/3) \cdot (2/3)^m$ and

$$\widetilde{\psi}_{x,m}\restriction_{V_m} = \mathbb{1}_{\{x\}} \quad \text{and} \quad \widetilde{\psi}_{x,m}\restriction_{M_e} \text{ harmonic}$$

Theorem (Approx. by metric graphs,[PS18b])

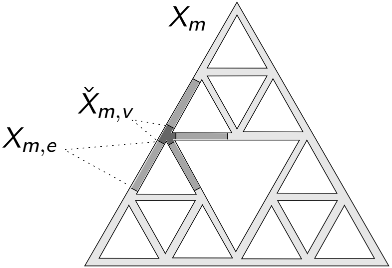
The energy form $\mathcal{E}$ on SG and the rescaled energy form $\tau_m \mathcal{E}_{M_m}$ on the associated metric graphs are δ_m-quasi-unitarily equivalent and

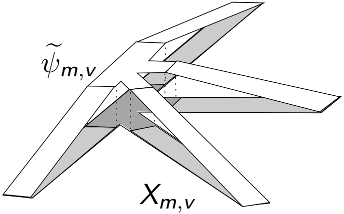
$$\delta_m \sim \frac{1}{5^{m/2}}$$

Main results: Graph-like manifold

A *graph-like manifold* is a Riemannian manifold of dimension $d \geq 2$ glued together from vertex neighbourhoods and edge neighbourhoods, respecting the structure of the graph

$$X_m = \underbrace{\bigcup_{v \in V_m} X_{m,v}}_{\text{vertex neighbourhoods}} \quad \cup \quad \underbrace{\bigcup_{e \in E_m} X_{m,e}}_{\text{edge neighbourhoods}}$$





Main results: Graph-like manifold

1. $\mathscr{H}_m := \mathsf{L}_2(X_m, \nu_m)$ with Riemannian measure ν and norm

$$\|u\|^2_{\mathscr{H}_m} = \int_{X_m} |u(x)|^2 \, \mathrm{d}\nu_m(x)$$

2. $(\mathcal{E}_{X_m}, \mathscr{H}^1_m)$, where $\mathscr{H}^1_m = \mathsf{H}^1(X_m, \nu_m)$ and

$$\mathcal{E}_{X_m}(u) = 3 \cdot \left(\frac{5}{4}\right)^m \int_{X_m} |\nabla u(x)|^2_x \, \mathrm{d}\nu_m(x)$$

where ∇ is the gradient and $|.|_x$ is the Riemannian metric.

Theorem (Approx. by graph-like manifolds)

The energy form $\mathcal{E}$ on SG and the rescaled energy form $\tau_m \mathcal{E}_{X_m}$ on the associated graph-like manifolds X_m are δ_m-quasi-unitarily equivalent where

$$\delta_m \sim \frac{1}{5^{m/3}}$$

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