Localization for Anderson Models on Tree Graphs

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Anderson model

- Schrödinger equation

$$\mathbf{i}\partial_t \varphi(t,x) = -\Delta\varphi(t,x) + V_\omega(x)\varphi(t,x);\; t\in\mathbb{R}_+, x\in\Gamma,$$
$$\varphi(0,x) = \varphi_0(x);\;\; x\in\Gamma$$

- E.g. $\Gamma = \mathbb{R}^d$, $\Gamma = \mathbb{Z}^d$, $\Gamma = (\mathcal{V},\mathcal{E})$ discrete or metric graph
- Unitary time evolution

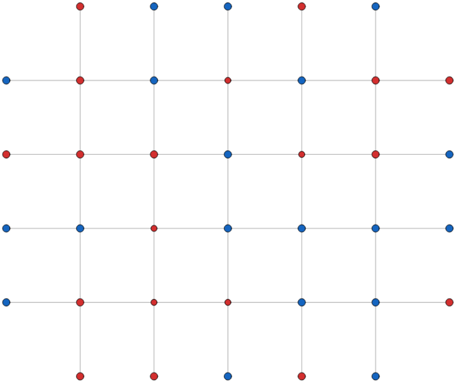
$$\varphi(t) = e^{-\mathbf{i}tH_\omega}\varphi_0$$

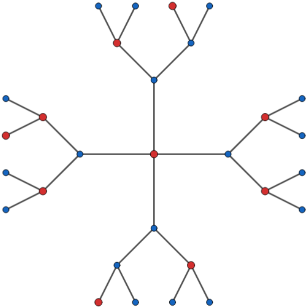
Goal: Prove Dynamical Localization (in some energy regions)

Solution $\varphi(t,\cdot)$ is localized whenever initial state $\varphi_0(\cdot)$ is localized, almost surely

The spectral localization (i.e. p.p. spectrum&exponentially decaying eigenfuncitons) is a prerequisite.

Anderson–Bernoulli model on $\mathbb{Z}^d$ and on a tree





$$\ell^2(\mathcal{V}) = \{\varphi : \mathcal{V} \to \mathbb{C} : \sum_{\nu \in \mathcal{V}} |\varphi(\nu)|^2 < \infty\}$$

$$H_\omega = \underbrace{-\Delta}_{\text{adj. matrix of } \mathcal{V}} + \underbrace{V_\omega}_{\text{mult. by } \omega_\nu \text{ i.i.d.}} \quad ; \quad H_\omega \in \mathcal{B}(\ell^2(\mathcal{V}))$$

RAGE Theorem

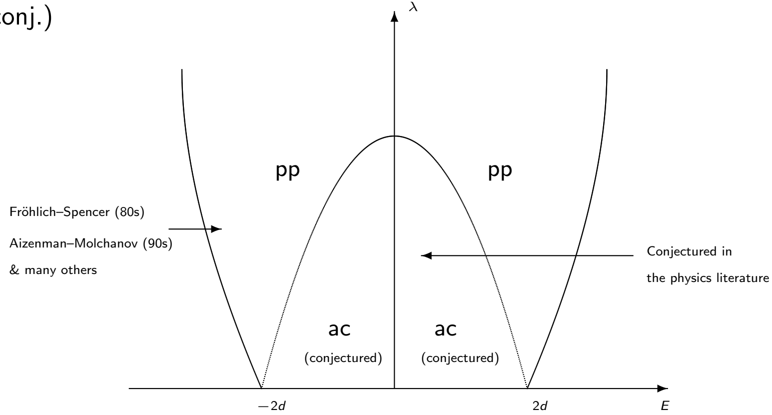
- $$\langle \varphi, f(H)\varphi \rangle_{\ell^2(\mathcal{V})} = \int_{\mathrm{spec}(H)} f(E) d\mu_\varphi(E)$$
- Spectral measure
$$d\mu_\varphi = d\mu_{\varphi,ac} + d\mu_{\varphi,sc} + d\mu_{\varphi,pp}$$
- $\sigma_\bullet(H) := \sigma\{H \restriction_{\ell^2(\mathcal{V})_\bullet}\}$, $\bullet \in$ {ac, sc, pp}.

Given an initial state $\varphi \in \ell^2(\mathcal{V})$ we have

- μ_φ is absolutely continuous then the particle "travels freely"
- μ_φ is pure point then the "dynamic is confined" to a compact set
- μ_φ is singular continuous the particle "escapes" compact sets in some time-average sense

Phase diagram for Anderson model on $\mathbb{Z}^d$

- $d = 1$: Localization for arbitrary strength of disorder (proved)
- $d = 2$: Localization for arbitrary strength of disorder (conjecture)
- $d > 2$: Localization near spectral edges (proved) & extended states for weak disorder (conj.)

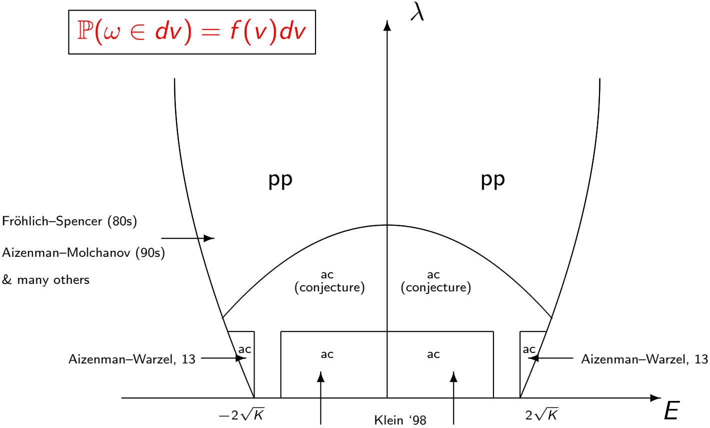


$$H_{\omega,\lambda} := -\Delta + \lambda V_\omega,\ \ \lambda\text{–strength of disorder}$$

$$\sigma(H_{\omega,\lambda}) = [-2d, 2d] + \lambda\, \mathrm{supp}(\mu),\ a.e.$$

Phase diagram for Anderson model on the discrete Bethe lattice

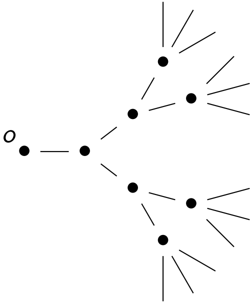
$\mathbb{P}(\omega \in dv) = f(v)dv$



$$H_{\omega,\lambda} := -\Delta + \lambda V_\omega, \ \lambda\text{–strength of disorder}$$

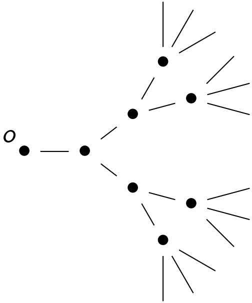
$$\sigma(H_{\omega,\lambda}) = [-2\sqrt{K}, 2\sqrt{K}] + \lambda\, \mathrm{supp}(\mu), \, a.e.$$

Random continuum models



- $b(v) = \omega_1(|v|), \quad \ell_e = \omega_2(gen(e)), \quad q(v) = \omega_3(|v|).$
- $\omega_{1,2,3}$ are sequences of i.i.d. random variables with **arbitrary** non-trivial distributions
- $\mathbb{H}_\omega$ denotes the Laplace operator on a random realization of the tree subject to random $\delta-$type vertex conditions.
- In particular, we consider Random branching model (RBM), Random length model (RLM), and Random Kirchhoff model (RKM).

Random discrete models



- $b(v) = \omega_1(|v|), \quad p(u,v) = \omega_2((u,v)), \quad q(v) = \omega_3(|v|).$
- $\omega_{1,2,3}$ are sequences of i.i.d. random variables with **arbitrary** non-trivial distributions

$$[\mathbb{J}_\omega f](u) = -\sum_{v\sim_\omega u} p_\omega(u,v) f(v), \quad f \in \ell^2(\mathcal{V}),\ v \in \mathcal{V},$$

or

$$[\mathbb{J}_\omega f](u) = \sum_{v\sim_\omega u} (q_\omega(u) f(u) - f(v)), \quad f \in \ell^2(\mathcal{V}),\ v \in \mathcal{V}.$$

Dynamical and spectral localization for metric trees

Theorem, [Damanik–Fillman–S, 2019]

There exists a discrete set $\mathfrak{D} \subseteq \mathbb{R}$ such that the following two assertions hold.

(i) The operator $\mathbb{H}_\omega$ exhibits Anderson localization at all energies outside of $\mathfrak{D}$. That is, almost surely, $\mathbb{H}_\omega$ has pure point spectrum and any eigenfunction of $\mathbb{H}_\omega$ corresponding to an energy $E \in \mathbb{R} \setminus \mathfrak{D}$ enjoys an exponential decay estimate of the form

$$|f(x)| \leq \frac{Ce^{-\lambda|x|}}{\sqrt{w_o(|x|)}}$$

with $C > 0$ and $\lambda > 0$, where $w_o(|x|)$ denotes the number of vertices in the generation of x, i.e., $w_o(|x|) = \#\{y \in \mathcal{V} : gen(y) = gen(x)\}$.

(ii) For every compact interval $I \in \mathbb{R} \setminus \mathfrak{D}$ there exists a set $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$ such that for every $p > 0$ and every compact set $\mathcal{K} \subset \Gamma_{b_\omega, \ell_\omega}$ one has

$$\sup_{t>0} \left\| |X|^p \chi_I(\mathbb{H}_\omega) e^{-it\mathbb{H}_\omega} \chi_{\mathcal{K}} \right\|_{L^2(\Gamma_{b_\omega, \ell_\omega})} < \infty, \ \omega \in \Omega^*,$$

where $\chi_I(\mathbb{H}_\omega)$ is the spectral projection corresponding to I.

Dynamical and spectral localization for metric trees

Theorem, [Damanik–Fillman–S, 2019]

There exists a set $\mathcal{D}$ of cardinality at most one such that the following assertions hold.

(i) The operator $\mathbb{J}_\omega$ exhibits Anderson localization at all energies outside of $\mathcal{D}$. That is, almost surely, $\mathbb{J}_\omega$ has pure point spectrum and any eigenfunction of $\mathbb{J}_\omega$ corresponding to an energy $E \in \mathbb{R} \setminus \mathcal{D}$ enjoys an exponential decay estimate of the form

$$|f(x)| \leq \frac{Ce^{-\lambda|x|}}{\sqrt{w_o(|x|)}}, x \in \mathcal{V},$$

where $C, \lambda > 0$ are constants.

Dynamical and spectral localization for discrete trees, contd.

(ii) For every compact interval $I \subset \mathbb{R} \setminus \mathcal{D}$ there exist $\Omega^* \subset \Omega$ with $\mu(\Omega^*) = 1$ and $\theta > 0$ such that for every $x, y \in \mathcal{V}$, $|x| \geq |y|$, $\omega \in \Omega^*$ one has

$$\sup_{t>0} |\langle \delta_x, \chi_I(\mathbb{J}_\omega) e^{-it\mathbb{J}_\omega} \delta_y \rangle_{\ell^2(\mathcal{V})}| \leq \frac{Ce^{-\theta \, \mathrm{dist}(x,y)}}{\sqrt{w_y(|x| - |y|)}},$$

for some $C = C(y, \omega, \theta) > 0$. In particular, for all $y \in \mathcal{V}$, $\omega \in \Omega^*$, $R > 0$ one has

$$\sum_{|x| \geq R} \sup_{t>0} |\langle \delta_x, \chi_I(\mathbb{J}_\omega) e^{-it\mathbb{J}_\omega} \delta_y \rangle_{\ell^2(\mathcal{V})}| \leq \gamma e^{-\kappa R},$$

for some $\kappa = \kappa(y) > 0$ and $\gamma = \gamma(y) > 0$.

- Naimark–Solomyak decomposition

$$\mathbb{H}_\omega = \mathbb{H}(b_\omega, \ell_\omega, q_\omega) = \bigoplus_{\varkappa=0}^{\infty} \bigoplus_{k=1}^{m(\varkappa)} H(T^\varkappa b_\omega, T^\varkappa \ell_\omega, T^\varkappa q_\omega),$$

- $H_\omega = H(T^\varkappa b_\omega, T^\varkappa \ell_\omega, T^\varkappa q_\omega)$ is the Laplace operator on the half-line $(t_\kappa, +\infty)$ subject to Dirichlet condition at t_κ and

$$\begin{cases} \sqrt{b_j} f(t_j^-) = f(t_j^+), & j \geq \varkappa \\ f'(t_j^-) + q_j f(t_j^-) = \sqrt{b_j} f'(t_j^+) & j \geq \varkappa, \end{cases}$$

Theorem, [Damanik–Fillman–S, 2019]

There exists a discrete set $\mathfrak{D} \subset \mathbb{R}$ such that for every compact interval $I \subseteq \mathbb{R} \setminus \mathfrak{D}$ and every $p > 0$, there exists $\widetilde{\Omega} \subset \Omega$ with $\mu(\widetilde{\Omega}) = 1$ such that

$$\sup_{t>0} \left\| |X|^p \chi_I(H_\omega) e^{-itH_\omega} \psi \right\|_{L^2(\mathbb{R}_+)} < \infty, \ \omega \in \widetilde{\Omega},$$

whenever $\psi \in L^2(\mathbb{R}_+)$ and $\psi(x) \underset{x\to\infty}{=} \mathcal{O}(e^{-\log^{22+\varepsilon} x}), \varepsilon > 0.$

- $$\mathbb{J}(b,p,q) \sim \bigoplus_{n=0}^{\infty} \bigoplus_{k=1}^{m(n)} J(T^n b, T^n p, T^n q),$$

$$J(b,p,q) := \begin{pmatrix} (b_0 p_0 + p_{-1}) q_0 & \sqrt{b_0} p_0 & 0 & \\ \sqrt{b_0} p_0 & (b_1 p_1 + p_0) q_1 & \sqrt{b_1} p_1 & \ddots \\ 0 & \sqrt{b_1} p_1 & \ddots & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Theorem, [Damanik–Fillman–S, 2019]

The operator J_ω exhibits Anderson localization at all energies outside of a set $\mathcal{D}$ of cardinality at most one.

Strategy of the Proof

- Lyapunov exponent. The cocycle is defined by

$$M_n^E(\omega) : \begin{bmatrix} u(0^+) \\ u'(0^+) \end{bmatrix} \mapsto \begin{bmatrix} u(t_n^+) \\ u'(t_n^+) \end{bmatrix}$$

where $-u'' = Eu$ and u satisfies the vertex conditions defining $\mathrm{dom}(H_\omega)$. This is a product of $\mathrm{SL}(2,\mathbb{R})$ random matrices. Lyapunov exponent for $E \in \mathbb{R}$

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\|; \ \mu - a.e.\ \omega$$

this limit exists and it is deterministic due to ergodicity.

- $L(E) > 0$ for $E \in \mathbb{R} \setminus \mathfrak{D}$(not easy!). Thus there is an exponentially decaying solution (by the Osceledets Theorem).

Anderson localizer's dream

For $\mu-$ a.e. ω one has

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\|; \text{ for all } E \in \mathbb{R}$$

Anderson localizer's dream

For $\mu-$ a.e. ω one has

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\|; \text{ for all } E \in \mathbb{R}$$

Gorodetski–Kleptsyn: For $\mu-$a.e. ω there exists E from a dense G_δ subset of the almost–sure spectrum such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\| < \limsup_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\|$$

Anderson localizer's dream revised

For $\mu-$ a.e. ω one has

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n^E(\omega)\|; \text{ for all } \textbf{generalized eigenvalues } E$$

Claim, [Damanik–Fillman–S, 2019]

The revised Anderson localizer's dream holds and it implies spectral localization. Furthermore, the dynamical localization holds.

Strategy of the Proof

- Fürstenberg's Theorem for $L(E) > 0$
- Elimination of double resonances (originally by Bourgain–Schlag for the doubling map model)
- Avalanche principle (due to Goldstein–Schlag)
- Semi-uniformly localized eigenfunctions
- Asymptotic formula for the number of centers of localizaiton in $[0, L]$ as $L \to \infty$
- Naimark–Solomyak/Breuer decomposition reversed