Spectral problems on star graphs

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(joint with V. Pivovarchik/N. Rozhenko and B.M. Brown/H. Langer)

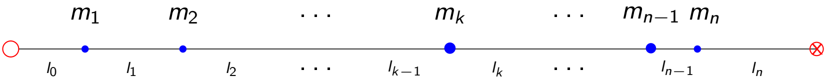
TU Graz, 26.02.2019

0. STIELTJES STRINGS

Stieltjes string: [Gantmakher, Kreĭn 1941/50]

elastic string of zero density bearing n point masses with $n \in \mathbb{N} \cup \{\infty\}$;
first $n < \infty$ (if $n = \infty$, accumulation only at one endpoint)

○ Dirichlet condition



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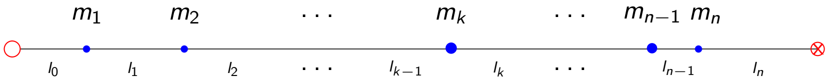

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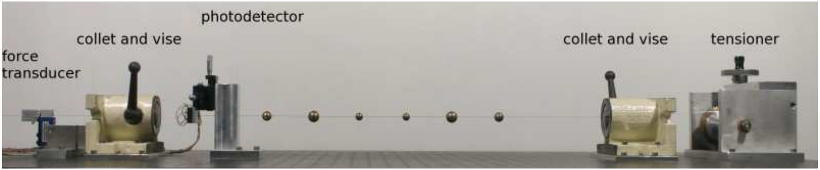
An experimental setup: [Cox, Embree, Hokanson SIAM Rev. 2012*]



*'One Can Hear the Composition of a String: Experiments with an Inverse Eigenvalue Problem'

Equations of motion:

$v_k(t), k=1,2,\ldots,n$: transverse displacements of k-th mass m_k at time t,

$k=0, n+1$: ——— || ——— of left/right end of string.

Lagrange equations:

$$\frac{v_k(t)-v_{k+1}(t)}{l_k}+\frac{v_k(t)-v_{k-1}(t)}{l_{k-1}}+m_k v_k''(t)=0, \quad k=1,2,\ldots,n.$$

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Boundary equations: Dirichlet at left end, Dirichlet/Neumann at right end

$$v_0(t) = 0, \quad v_{n+1}(t) = 0 \qquad \text{(DD)}$$

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Separation of variables: $v_k(t)=u_k\,\mathrm{e}^{\mathrm{i}\lambda t}$, $k=0,1,\ldots,n+1$ $\rightsquigarrow$

$$\frac{u_k-u_{k+1}}{l_k}+\frac{u_k-u_{k-1}}{l_{k-1}}-m_k\lambda^2 u_k=0 \quad k=1,2,..,n,$$

$$\begin{cases} u_0=0, \quad u_{n+1}=0, & \text{(DD)},\\ u_0=0, \quad u_{n+1}-u_n=0, & \text{(DN)} \end{cases}$$

Spectra:

[GK50, Anhang II] Eine bemerkenswerte Aufgabe für eine Perlenschnur ....

$$\frac{u_k - u_{k+1}}{l_k} + \frac{u_k - u_{k-1}}{l_{k-1}} - m_k \lambda^2 u_k = 0, \quad k = 1, 2, .., n,$$

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$z := \lambda^2$: Solve for u_{k+1} $\rightsquigarrow$ two-term recurrence relation for

$$u_{k+1} = R_{2k}(z) u_1, \quad \frac{u_{k+1} - u_k}{l_k} = R_{2k-1}(z) u_1, \quad k = 0, 1, \ldots, n,$$

with polynomials R_{2k-1}, R_{2k}, $k = 1, 2, \ldots, 2n$, of degree k satisfying

$$\begin{aligned} R_{2k-1}(z) &= -m_k z R_{2k-2}(z) + R_{2k-3}(z), \\ R_{2k}(z) &= \qquad l_k R_{2k-1}(z) + R_{2k-2}(z), \end{aligned} \qquad k = 1, 2, \ldots, n.$$

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(D...) $u_0 = 0 \implies R_{-1}(z) = \frac{1}{l_0}$, $R_0(z) = 1$ (start of recurrence, $k = 0$)

(DD) $u_{n+1} = 0 \implies R_{2n}(\lambda^2) = 0$ Dirichlet eigenvalues

(DN) $u_{n+1} - u_n = 0 \implies R_{2n-1}(\lambda^2) = 0$ Neumann eigenvalues

[GK50, Anhang II] .... und über Stieltjessche Kettenbrüche

Dirichlet and Neumann eigenvalues are zeros and poles, respectively, of

$$\varphi(z) := \frac{R_{2n}(z)}{R_{2n-1}(z)} = l_n + \frac{1}{\frac{R_{2n-1}(z)}{R_{2n-2}(z)}} = l_n + \frac{1}{-m_n z + \frac{1}{\frac{R_{2n-2}(z)}{R_{2n-3}(z)}}}$$

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$\varphi(0) = l_0 + l_1 + \cdots + l_n =: l$ (length of string)

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[Stieltjes 1894]

RECHERCHES

SUR

LES FRACTIONS CONTINUES,

PAR M. T.-J. STIELTJES,

Professeur à la Faculté des Sciences de Toulouse.

INTRODUCTION.

L'objet de ce travail est l'étude de la fraction continue

$$(1) \qquad \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \ddots + \cfrac{1}{a_{2n} + \cfrac{1}{a_{2n+1} z + \ddots}}}}},$$

dans laquelle les a_i sont des nombres réels et positifs, tandis que z est une variable qui peut prendre toutes les valeurs réelles ou imaginaires.

En désignant par $\frac{P_n(z)}{Q_n(z)}$ la $n^{\text{ième}}$ réduite qui ne dépend que des n premiers coefficients a_i, nous chercherons dans quels cas cette réduite tend vers une limite pour $n = \infty$ et nous aurons à approfondir la nature de cette limite considérée comme une fonction de z.

Nous allons résumer le résultat le plus essentiel de cette étude. Il y a deux cas à distinguer.

Premier cas. — La série $\sum_1^\infty a_n$ est convergente.

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$\varphi(0) = l_0 + l_1 + \cdots + l_n =: l$ (length of string)

DEF. $z \mapsto f(z)$ is called *S-function* (Stieltjes function) $:\Longleftrightarrow$

- f Nevanlinna function
  (i.e. analytic in $\mathbb{C} \setminus [0, \infty)$, $\mathrm{Im} z \cdot \mathrm{Im} f(z) \geq 0$ for $\mathrm{Im} z \neq 0$),
- $f(z) > 0$, $z \in (-\infty, 0)$,

and S_0*-function* if

- 0 is no pole of f.

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PROP. φ is an S_0-function.

LEMMA S. **i)** f is a rational S_0-function $\Longrightarrow$ for some $p \in \mathbb{N}$

$$f(z) = a_0 + \cfrac{1}{-b_1 z + \cfrac{1}{a_1 + \cfrac{1}{-b_2 z + \cdots + \cfrac{1}{a_{p-1} + \cfrac{1}{-b_p z + \cfrac{1}{a_p}}}}}} \qquad \text{(CFE)}$$

with (unique) $a_0 = \lim_{z \to \pm\infty} f(z) \geq 0$, a_k, $b_k > 0$, $k = 1, 2, \ldots, p$, and strictly interlacing zeros α_k and poles β_k,

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{p-1} < \alpha_p \ (< \beta_p \text{ if } a_0 > 0). \qquad (*)$$

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For φ: $p = n$ (number of masses),

$$a_0 = l_n > 0, \quad a_k = l_{n-k}, \quad b_k = m_{n-k+1}, \quad k = 1, 2, \ldots, n,$$
$$f(0) = a_0 + a_1 + \cdots + a_p = l_0 + l_1 + \cdots + l_n = l,$$

where l is the total length of the Stieltjes string.

Inverse Problem

Given: $0 < \lambda_1 < \zeta_1 < \ldots < \lambda_n < \zeta_n$ and $l > 0$.

Find: masses m_k and lengths l_k between them so that $(\zeta_k)_1^n$ and $(\lambda_k)_1^n$ are Dirichlet and Neumann eigenvalues of corr. Stieltjes string of length l!

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Solution: Introduce

$$\Psi(z) := l \cdot \frac{\prod_{k=1}^n \left(1 - \frac{z}{\zeta_k^2}\right)}{\prod_{k=1}^n \left(1 - \frac{z}{\lambda_k^2}\right)} \qquad (\rightsquigarrow \Psi(0) = l);$$

Lemma S $\implies$ Ψ is S_0-function and admits a unique CFE $\rightsquigarrow$

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with $a_0 = \lim_{z\to\pm\infty} \Psi(z) > 0$, $a_0 + a_1 + \cdots + a_n = \Psi(0) = l$. Set

$$l_n := a_0, \qquad l_k = a_{n-k}, \quad m_k = b_{n-k+1}, \qquad k = 1, 2, \ldots, n.$$

I. STAR GRAPH OF STIELTJIES STRINGS

A star-shaped vibrating system:

Tritare$^{\mathrm{TM}}$,

guitar-like instrument using 6 Y-shaped networks instead of 6 simple strings;

[Gaudet/Léger '03] A new family of stringed musical instruments, see

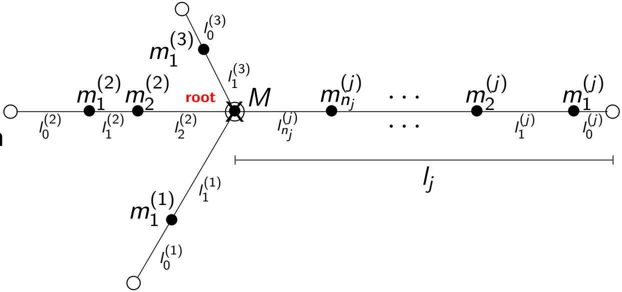
`http://acoustics.org/pressroom/httpdocs/151st/Leger.html`

to hear how it may sound.

Star graph of Stieltjes strings with finite number of masses:

○ Dirichlet condition

× Neumann condition



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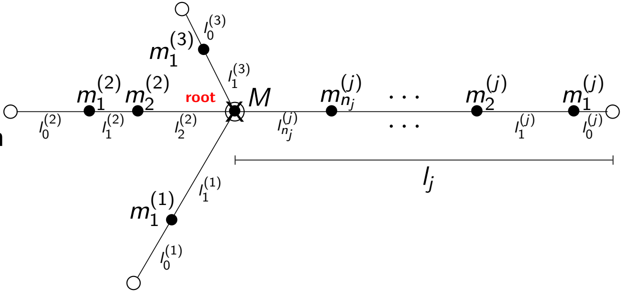

Notation. • number of Stieltjes strings $q \geq 2$;

- string j has n_j masses and length l_j;
- masses $m_k^{(j)}$ and lengths between $l_k^{(j)}$ on string j (numbered from ext.);
- mass $M \geq 0$ at central vertex.

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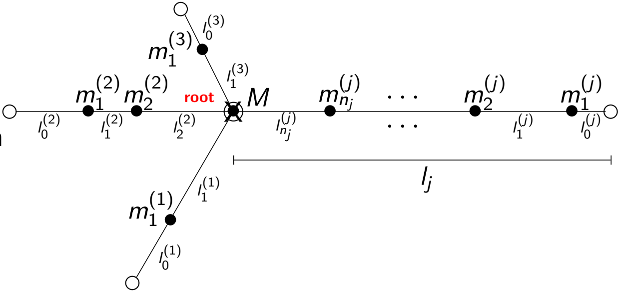

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Two problems. Dirichlet conditions at pendant vertices + continuity and

Dirichlet condition at central vertex for (D1),

Neumann condition at central vertex for (N1).

Problem (D1) ...... decouples into q Stieltjes strings with (DD):

$$\left.\begin{array}{l} \dfrac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\dfrac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}\lambda^2 u_k^{(j)}=0, \quad k=1,2,..,n_j, \\ u_{n_j+1}^{(j)}=0, \quad \text{(Dirichlet at central vertex)} \\ u_0^{(j)}=0, \quad \text{(Dirichlet at pendant vertices)} \end{array}\right\} \; j=1,2,\dots,q.$$

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Problem (N1):

$$\dfrac{u_k^{(j)}-u_{k+1}^{(j)}}{l_k^{(j)}}+\dfrac{u_k^{(j)}-u_{k-1}^{(j)}}{l_{k-1}^{(j)}}-m_k^{(j)}\lambda^2 u_k^{(j)}=0, \quad k=1,2,..,n_j, \quad j=1,2,..,q,$$

$$u_{n_1+1}^{(1)}=u_{n_2+1}^{(2)}=\ldots=u_{n_q+1}^{(q)}, \qquad \text{(continuity at central vertex)}$$

$$\sum_{j=1}^{q}\dfrac{u_{n_j+1}^{(j)}-u_{n_j}^{(j)}}{l_{n_j}^{(j)}}=M\lambda^2 u_{n_1+1}^{(1)}, \qquad \text{(Neumann at central vertex)}$$

$$u_0^{(j)}=0, \quad j=1,2,\ldots,q. \qquad \text{(Dirichlet at pendant vertices)}$$

Direct spectral problem:

THM. 1. The Dirichlet and Neumann eigenvalues of (D1) and (N1) are the zeros and poles of

$$\phi_q(z) := \frac{\phi_{D,q}(z)}{\phi_{N,q}(z)} = \frac{1}{\sum_{j=1}^{q} \frac{1}{\varphi_j(z)} - Mz}$$

which, after cancellation of common factors (if any) in the numerator and in the denominator, becomes an S_0-function.

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- number of Dirichlet eigenvalues ζ_k: $n = n_1 + \cdots + n_q$;
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In the sequel: $M > 0$.

THM. 2. $\{\zeta_k^2\}_{k=1}^n$ and $\{\lambda_k^2\}_{k=1}^{n+1}$ satisfy:

1) $0 < \lambda_1 < \zeta_1 \leq \lambda_2 \leq \ldots \leq \zeta_{n-1} \leq \lambda_n \leq \zeta_n < \lambda_{n+1}$; $(\lambda_{n+1} := \infty \text{ if } M = 0)$
2) $\zeta_{k-1} = \lambda_k$ if and only if $\lambda_k = \zeta_k$, $k = 2, 3, \ldots, n$;
3) the multiplicity of λ_k does not exceed $q - 1$.

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3) the multiplicity of λ_k does not exceed $q - 1$.

COR. One of two disjoint neighbouring Neumann eigenvalues is simple.

REM. λ_k is a monotonically decreasing function of $M \in [0, \infty)$ and

$$\lambda_1 \xrightarrow{M\to\infty} 0, \quad \lambda_{k+1} \xrightarrow{M\to\infty} \zeta_k, \quad k = 1, 2, \ldots, n.$$

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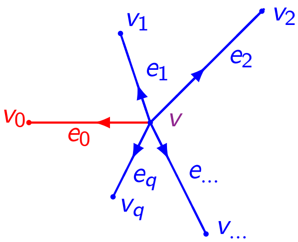
REM. Analogous results when root is at pendant vertex.

II. A REDUCTION RESULT



Star graph $\mathcal{G}$ with distinguished edge e_0

T_0, $T_1, \dots, T_q$, symmetric linear relations with equal defect numbers 1, e.g. induced by differential operators on e_0, $e_1, \dots, e_q$,

[Simonov/Woracek '14]

here joined by an arbitrary selfadjoint 'matching condition' $\rightsquigarrow$ $\mathbf{T}_{\mathcal{A},\mathcal{B}}$.

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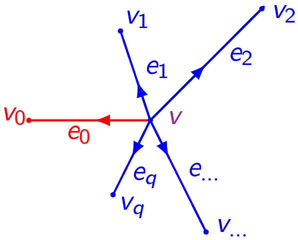

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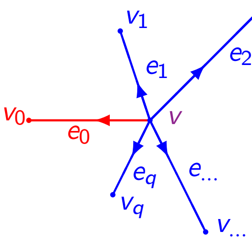

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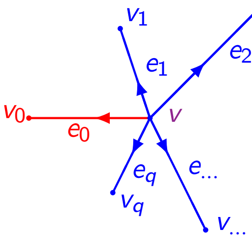

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ANSWERS: YES, after a two step reduction ...

Strategy of two step reduction:

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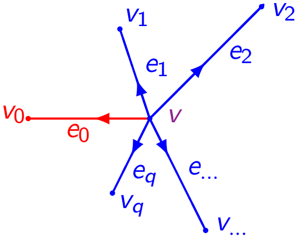

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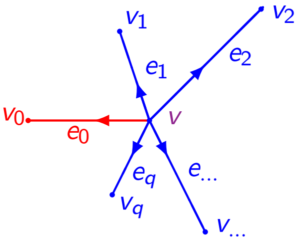

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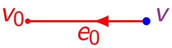

$\Gamma_{01} g_0 + n_0(z) \Gamma_{02} g_0 = 0$

Edge e_0 with z-depending boundary condition at v

Step 1: Reduction to T_0^* with z-depending boundary condition

$\mathbf{T} = \bigoplus_{j=0}^{q} T_j$ with equal defect numbers $q+1$,

$(\mathbb{C}, \Gamma_{j1}, \Gamma_{j2})$, $(\mathbb{C}^{q+1}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2)$ boundary triplets for T_j^*, $\mathbf{T}^*$,

φ_j, $\mathcal{M} = \operatorname{diag}(\varphi_0, \varphi_1, \dots, \varphi_q)$ corr. Weyl-Titchmarsh functions,

$\mathbf{T}_{\mathcal{A},\mathcal{B}}$ restriction of $\mathbf{T}^*$ by $\mathcal{A}\,\mathbf{\Gamma}_1 \mathbf{y} + \mathcal{B}\,\mathbf{\Gamma}_2 \mathbf{y} = 0$ (MC):

$$n_0(z) := \frac{1}{\left((\mathcal{A}\mathcal{M}_q(z) - \mathcal{B})^{-1} \mathcal{A}\epsilon_0, \epsilon_0\right)_{\mathbb{C}^{q+1}}}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

$\mathcal{M}_q = \operatorname{diag}(0, \varphi_1, \dots, \varphi_q)$, $\epsilon_0 = (1, 0, \dots, 0)$, and

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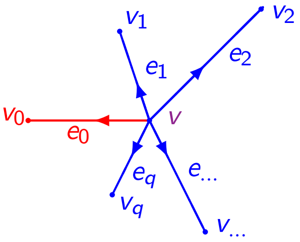

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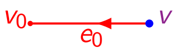

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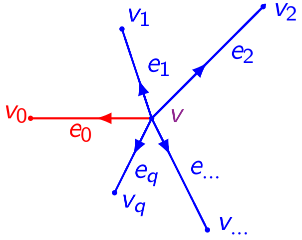
Strategy of two step reduction:



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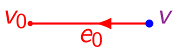

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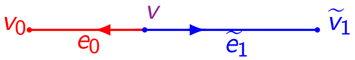
Edge e_0 with z-depending boundary condition at v



Reduced path graph $\widetilde{\mathcal{G}}$

Step 1: Reduction to T_0^* with z-depending boundary condition

Step 2: Replace z-depending boundary condition by new system $\widetilde{T}_1$,

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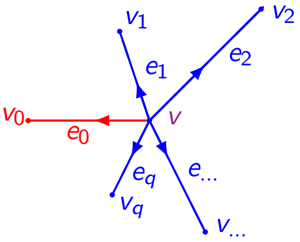

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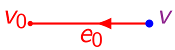

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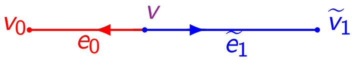

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Step 1: Reduction to T_0^* with z-depending boundary condition

Step 2: Replace z-depending boundary condition by new system $\widetilde{T}_1$,

by applying various inverse results to obtain

[Langer/Textorius '77] $\rightsquigarrow$ symmetric lin. relation $\widetilde{T}_1$,

[De Branges '62] $\rightsquigarrow$ trace-normed canonical system $\widetilde{T}_1$ on $\widetilde{e}_1 = [0, \infty)$,

[Kreĭn '68] $\rightsquigarrow$ if n_0 Stieltjes function: Krein string $\widetilde{T}_1$ on $\widetilde{e}_1$,

joined at v with standard matching conditions.

Special case: Robin type matching conditions

$$\mathcal{A}_R\,\mathbf{\Gamma}_1\mathbf{y} - \frac{1}{\tau}\mathcal{B}_R\,\mathbf{\Gamma}_2\mathbf{y} = 0 \qquad \text{(RtMC)}$$

($\tau = \infty$: Dirichlet conditions; $\tau = 0$: standard matching conditions); then

$$n_0(z) = \frac{1}{\tau + \frac{1}{\varphi_1(z)} + \frac{1}{\varphi_2(z)} + \cdots + \frac{1}{\varphi_q(z)}}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

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THM 4. Let all edges $e_j = [0, \ell_j]$, $j = 0, 1, ..., q$, be Stieltjes strings and let φ_j, $j = 1, 2, ..., q$, be the Titchmarsh-Weyl functions corr. to Neumann condition at $v \, (= 0)$. If

$$\tau \geq -\frac{1}{\varphi_1(0-)} - \frac{1}{\varphi_2(0-)} - \cdots - \frac{1}{\varphi_q(0-)}, \qquad (*)$$

there exists a Stieltjes string $\widetilde{e}_1$ joined to e_0 by standard matching conditions such that the resolvents for the *star graph* $\mathcal{G}$ with $q+1$ edges $e_0, e_1, ..., e_q$ and for the *path graph* $\widetilde{\mathcal{G}}$ with 2 edges e_0, $\widetilde{e}_1$ "coincide on e_0".

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Rem. $(*) \implies n_0$ Stieltjes function.

Rem. The Stieltjes string $\widetilde{e}_1$ with its masses $\widetilde{m}_k$, $k=1,2,\ldots,\widetilde{n}$, and its lengths $\widetilde{l}_k$, $k=0,1,\ldots,\widetilde{n}$, $\widetilde{n}\in\mathbb{N}\cup\{\infty\}$, is obtained from the CFE of n_0,

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THANK YOU!