Boundary representations of λ-harmonic and polyharmonic functions on trees

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joint work with M. A. Picardello

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Introduction

- Polyharmonic function of order n on a Euclidean domain D:

$$f : D \to \mathbb{C} \quad \text{such that} \quad \Delta^n f = 0\,.$$

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  [Gazzola, Grunau and Sweers, 2010]

- [Almansi, 1899]: if $D \subset \mathbb{R}^2 \equiv \mathbb{C}$ is starlike w.r.to origin, then

$$f(z) = \sum_{r=0}^{n-1} |z|^{2r}\, h_r(z) \quad \text{with } h_r \text{ harmonic.}$$

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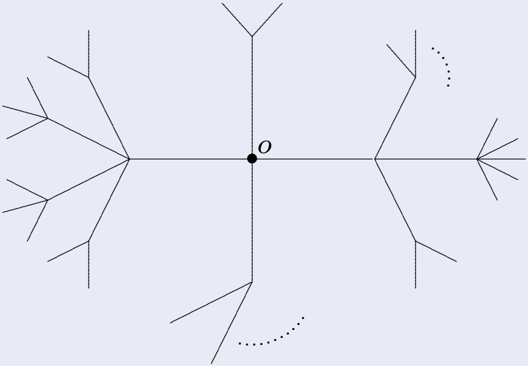
- If $D = \{zx + \mathrm{i}\ y \in \mathbb{C} : |z| < 1\}$ and
$$P(z,\xi) = \frac{1 - |z|^2}{|\xi - z|^2} \qquad (z \in D\,,\ \xi \in \partial D)$$
is the Poisson kernel, then Alamansi's representation can be rewritten as
$$f(z) = \sum_{r=0}^{n-1} \int_{\partial D} |z|^{2r}\, P(z,\xi)\, d\nu_r(\xi)\,,$$
where $\nu_0, \dots, \nu_{n-1}$ are analytic functionals (= certain distributions) on ∂D [Helgason, 1974];
Borel measures in the classical case $h_r \geq 0$.
- Discrete analogues ?

- *T* a countable tree with no leaves, vertices may have infinitely many neighbours.



Transition operators on trees

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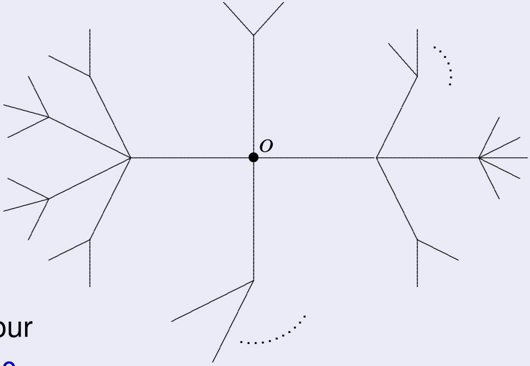

- $P = \big(p(x,y)\big)_{x,y \in T}$ stochastic transition matrix of nearest neighbour random walk: $p(x,y) > 0 \Leftrightarrow x \sim y$.

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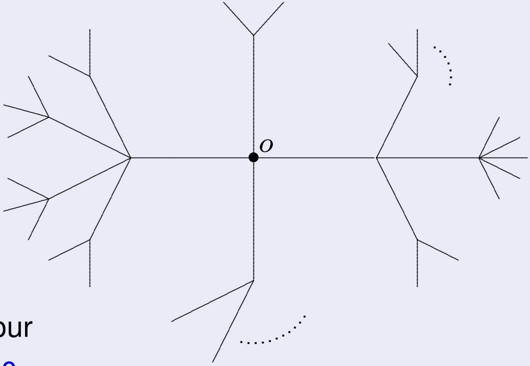

- $P = \big(p(x,y)\big)_{x,y\in T}$ stochastic transition matrix of nearest neighbour random walk: $p(x,y) > 0 \Leftrightarrow x \sim y$.

- Analogue of $-\Delta$ is $I - P$, where

$$Pf(x) = \sum_{y\sim x} p(x,y) f(y) \quad \text{for } f : T \to \mathbb{C} \quad \text{(abs. convergent).}$$

- A λ-harmonic function $(\lambda \in \mathbb{C})$ is a function $h : T \to \mathbb{C}$ with

$$Ph = \lambda \cdot h\,.$$

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$$\rho(P) = \limsup\, p^{(n)}(x,y)^{1/n}\,.$$
- Want $\lambda \in \mathrm{res}(P) = \mathbb{C} \setminus \mathrm{spec}(P)$ (possibly also $\lambda = \pm\rho(P)$.)

- Resolvent = Green function

$$G(x,y|\lambda) = (\lambda \cdot I - P)^{-1}\mathbf{1}_y(x)$$

for $|\lambda| > \rho(P)$

$$= \sum_{n=0}^{\infty} p^{(n)}(x,y)\Big/\lambda^{n+1}\,.$$

Satisfies $PG(\cdot,y|\lambda) = \lambda \cdot G(\cdot,y|\lambda) - \mathbf{1}_y\,.$

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$$\lambda \in \text{res}^*(P) = \{\lambda \in \text{res}(P) : G(x,x|\lambda) \neq 0 \ \forall\, x \in T\}$$

$$\supset \{\lambda \in \mathbb{C} : |\lambda| > \rho(P)\}$$

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- extends to the boundary ∂T in the 2^{nd} variable.

Boundary integral representation

Theorem [Picardello and W, 2018] For $\lambda \in \text{res}^*(P)$, every λ-harmonic function h on T has a unique integral representation

$$h(x) = \int_{\partial T} K(x, \xi|\lambda)\, d\nu(\xi),$$

where ν is a (strong) distribution on ∂T.

(Borel measure when $\lambda > \rho(P)$ and $h > 0$.)

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Analogue of Poisson integral representation of harmonic functions on the unit disk.

Generalizes different previous more restricted variants by [Cartier, 1972], [Cartwright, Soardi and Woess, 1993], [Figà-Talamanca and Steger, 1994], [Woess, 2009]

Geometry of the boundary

- Every boundary point ξ is represented by a geodesic ray starting from the root o,

$$\xi = [o = x_0\,, x_1\,, x_2\,, \dots] \quad \text{with} \quad x_n \sim x_{n-1}\,,\; x_n \neq x_m \quad \text{when} \quad m \neq n.$$

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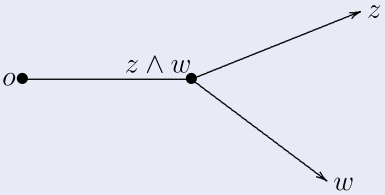
- Confluent of $w, z \in \widehat{T} := T \cup \partial T$ ($z \neq w$) : last common vertex on geodesic paths from o to w, resp. z.



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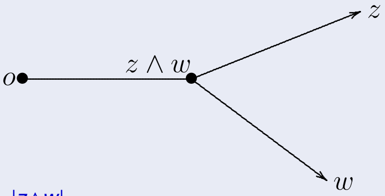

- New metric on $\widehat{T}$: $\theta(z,w) = \begin{cases} 2^{-|z \wedge w|}\,, & z \neq w \\ 0\,, & z = w. \end{cases}$

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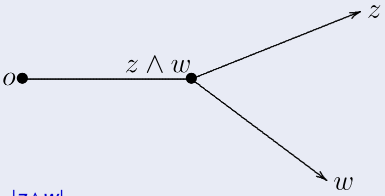

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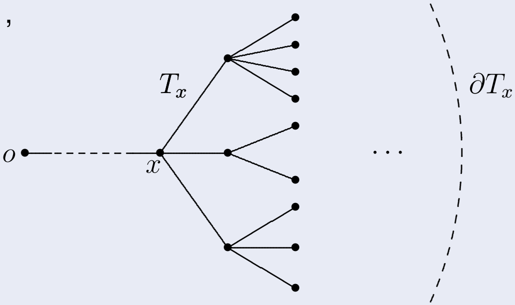
- T discrete and dense in $\widehat{T}$.
  $\widehat{T}$ compact only when T is locally finite.

Geometry of the boundary

- The branch T_x at $x \in T$,
  its boundary arc ∂T_x,
  and $\widehat{T}_x = T_x \cup \partial T_x$
  (open and closed !).
  Note that $T_o = T$.



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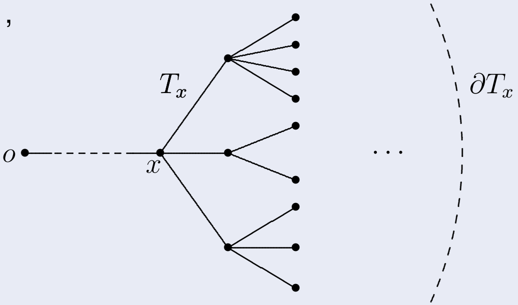

- For $\xi \in \partial T$,
  $\xi = [o = x_0, x_1, x_2, \dots]$,
  the sets $\widehat{T}_{x_n}$ form a neighbourhood basis.

- A (strong) distribution on $\mathcal{F}_o = \{\partial T_x : x \in T\}$ is a set function $\nu : \mathcal{F}_o \to \mathbb{C}$ such that for every $x \in T$,

$$\nu(\partial T_x) = \sum_{y : y^- = x} \nu(\partial T_y) \qquad \text{(absolutely convergent).}$$

Here, y^- = neighbour of y closer to o.

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- Locally constant function $f : T \to \mathbb{C}$: $\{x \sim y : f(x) \neq f(y)\}$ is finite. Extends continuously to $\widehat{T}$.
  Locally constant function φ on ∂T: trace of a l.c. function on T.

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$$\int_{\partial T} \varphi \, d\nu := \sum_{z \in \varphi(T)} z \cdot \nu[\varphi = z].$$

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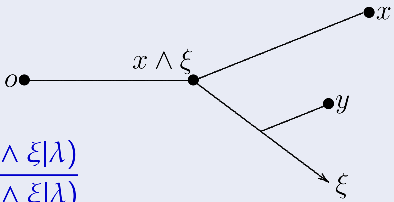

$$\Rightarrow \quad K(x,\xi|\lambda) = \lim_{y\to\xi} \frac{G(x,y|\lambda)}{G(o,y|\lambda)} = \frac{F(x,x\wedge\xi|\lambda)}{F(o,x\wedge\xi|\lambda)}$$

Boundary integral representation (restated)

Theorem [Picardello and W, 2018] For $\lambda \in \mathrm{res}^*(P)$, every λ-harmonic function h on T has a unique integral representation

$$h(x) = \int_{\partial T} K(x,\xi|\lambda)\, d\nu(\xi)\,,$$

where ν is the (strong) distribution on ∂T given by $\nu(\partial T) = h(o)$ and, for $x \neq o$,

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Theorem [Figà-Talamanca and Steger, 1994], [Picardello and W, 2018]
If P is invariant under a group Γ acting transitively on T
(i.e. $p(\gamma x, \gamma y) = p(x, y)\ \forall\ \gamma \in \Gamma$) then

$$\mathrm{res}(P) \setminus \{0\} \subset \mathrm{res}^*(P) \subset \mathrm{res}(P).$$

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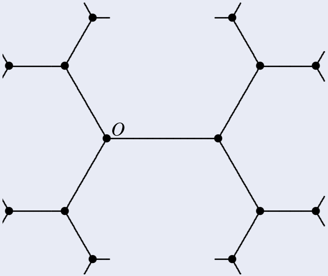
- Previous work (for $\lambda = 1$): [Cohen, Colonna, Gowrisankaran and Singman, 2002], in particular for regular tree of degree $q + 1$, and $p(x, y) = 1/(q + 1)$ for $x \sim y$.



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- A "basis" for space of λ-harmonic functions ($\lambda \in \mathrm{res}^*(P)$) is given by the Martin kernels

$$x \mapsto K(x, \xi|\lambda) = \frac{G(x, x \wedge \xi|\lambda)}{G(o, x \wedge \xi|\lambda)}, \quad \xi \in \partial T.$$

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- “Basis” for space of λ-polyharmonic functions ? Simple new idea: differentiate with respect to λ ! $\quad P\,K(\cdot,\xi|\lambda) = \lambda\,K(\cdot,\xi|\lambda) \quad \Rightarrow$

$$P\,\frac{d^r}{d\lambda^r}K(\cdot,\xi|\lambda) = \lambda\,\frac{d^r}{d\lambda^r}K(\cdot,\xi|\lambda) + r\,\frac{d^{r-1}}{d\lambda^{r-1}}K(\cdot,\xi|\lambda)$$

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- Set $\quad K_r(x,\xi|\lambda) = \dfrac{(-1)^r}{r!}\,\dfrac{d^r}{d\lambda^r}K(\cdot,\xi|\lambda)$.

  Then $x \mapsto K_r(x,\xi|\lambda)$ is λ-polyharmonic of order $r+1$.

  One deduces

Theorem [Picardello and W, 2018] For $\lambda \in \mathrm{res}^*(P)$, every λ-polyharmonic function f of order n on T has a unique representation

$$f(x) = \sum_{r=0}^{n-1} \int_{\partial T} K_r(x, \xi | \lambda) \, d\nu_r(\xi) \, ,$$

where where $\nu_0, \ldots, \nu_{r-1}$ are (strong) distributions on ∂T.

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  (ii) boundary integral representation of all λ-harmonic function with respect to suitable distributions (functionals on suitable space of functions on the boundary).

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- Continuous analogues ?
  Yes, hyperbolic Laplacian on Poincaré disk (ongoing work).

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- Poincaré disk = unit disk $D = \{z = x + \mathrm{i}\, y \in \mathbb{C} : |z| < 1\}$ with hyperbolic length element and metric

$$d_h s = \frac{2\sqrt{dx^2 + dy^2}}{1 - |z|^2} \quad \text{and} \quad d_h(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}\,.$$

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- The hyperbolic Laplace(-Beltrami) operator in $z = x + \mathrm{i}\, y$ is

$$\Delta_h = \frac{(1 - |z|^2)^2}{4}\left(\partial_x^2 + \partial_y^2\right).$$

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$$d_h s = \frac{2\sqrt{dx^2 + dy^2}}{1 - |z|^2} \quad \text{and} \quad d_h(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}.$$

- The hyperbolic Laplace(-Beltrami) operator in $z = x + \mathrm{i}\, y$ is

$$\Delta_h = \frac{(1 - |z|^2)^2}{4} \left(\partial_x^2 + \partial_y^2\right).$$

- Poisson kernel $P(z, \xi) = \dfrac{1 - |z|^2}{|\xi - z|^2} \qquad (z \in D\,,\ \xi \in \partial D)$

$$= e^{-\mathfrak{h}(z,\xi)} \quad \text{with} \quad \mathfrak{h}(z, \xi) = \lim_{w \to \xi} \Big(d_h(w, z) - d_h(w, 0) \Big),$$

the Busemann function.

λ-polyharmonic functions on the Poincaré disk

- λ-polyharmonic function of order n: function $f : D \to \mathbb{C}$ with
$$(\lambda \cdot I - \Delta_h)^n f = 0 \,.$$
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Theorem [Helgason, 1974] For $\lambda \in \mathbb{C} \setminus (-\infty\,, -1/4]$, every λ-harmonic function h on D has a unique integral representation

$$h(z) = \int_{\partial D} P(z,\xi)^{t(\lambda)}\, d\nu(\xi)\,,$$

where ν is a an analytic functional on ∂D.

(Borel measure when $\lambda > -1/4$ and $h > 0$.)

Theorem [W, 2018] For $\lambda \in \mathbb{C} \setminus (-\infty, -1/4]$, every λ-polyharmonic function f of order n on D has a unique representation

$$f(z) = \sum_{r=0}^{n-1} \int_{\partial D} \mathfrak{h}(z,\xi)^r \, P(z,\xi)^{t(\lambda)} \, d\nu_r(\xi)\,,$$

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Compare with Almansi's theorem ($\lambda = 1$, Euclidean disk)

$$f(z) = \sum_{r=0}^{n-1} \int_{\partial D} |z|^{2r} \, P(z,\xi) \, d\nu_r(\xi)\,,$$

- For any open annulus A_δ containing ∂D, let $\mathcal{H}(A_\delta)$ be the space of holomorphic functions on A_δ.

  Topology: uniform convergence on compact sets.

Analytic functionals on the unit circle

Graz University of Technology

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- Analytic functionals on ∂D: topological dual space of $\mathcal{H}(\partial D)$ (continuous linear functionals).