Weak localization results for TE-modes

Ian Wood

University of Kent

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joint work with B.M. Brown (Cardiff), V. Hoang (San Antonio), M. Plum (Karlsruhe) and M. Radosz (San Antonio)

Waveguide

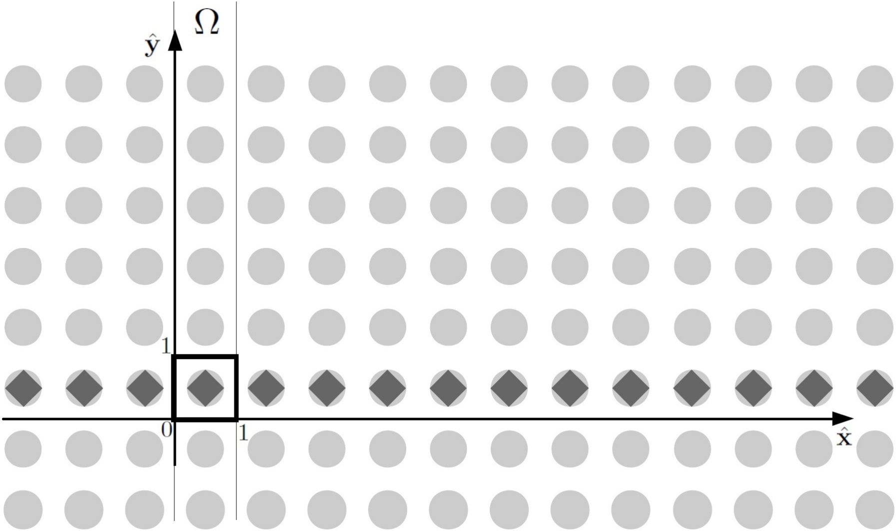

Figure: Illustration of the line defect and the strip $\Omega = (0,1) \times \mathbb{R}$.

Periodic Problem & Floquet Transform I

Consideration of TE-modes leads to the spectral problem for the selfadjoint operator L_0 acting on $L^2(\mathbb{R}^2)$ given by

$$L_0 u = -\mathrm{div}\ \frac{1}{\varepsilon_0}\nabla u,$$

where $\varepsilon_0(x, y) \geq c > 0$ is bounded and 1-periodic in both x and y.

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where $\varepsilon_0(x,y) \geq c > 0$ is bounded and 1-periodic in both x and y. Floquet transform U in the x-direction, gives a family of problems:

$$-\mathrm{div}\ \frac{1}{\varepsilon_0}\nabla u = \lambda u \quad \text{in } \Omega := (0,1)\times\mathbb{R}$$

with quasiperiodic boundary conditions

$$u(1,y) = e^{ik_x}u(0,y) \quad \text{and} \quad \frac{\partial u}{\partial x}(1,y) = e^{ik_x}\frac{\partial u}{\partial x}(0,y) \tag{1}$$

for $k_x \in B := [-\pi, \pi]$.

Periodic Problem & Floquet Transform II

Let $L_0(k_x)$ be the operator in $L^2(\Omega)$ given by

$$L_0(k_x)u = -\mathrm{div}\ \frac{1}{\varepsilon_0}\nabla u$$

subject to the quasi-periodic boundary conditions (1).

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subject to the quasi-periodic boundary conditions (1). Then

$$L_0 = \int_B^{\oplus} L_0(k_x)\ dk_x \quad \text{and} \quad \sigma(L_0) = \overline{\bigcup_{k_x \in B} \sigma(L_0(k_x))}.$$

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For each k_x, another Floquet transform in the y-direction gives operators $L_0(k_x, k)$, $k \in B$, on $L^2([0,1]^2)$ subject to qp-bcs in both x and y.
For the spectrum, we have

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$$\sigma(L_0(k_x)) = \overline{\bigcup_{k \in B} \sigma(L_0(k_x, k))} = \overline{\bigcup_n \left(\bigcup_{k \in B} \lambda_n(k_x, k) \right)}.$$

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Any gap in the spectrum of L_0 is due to gaps in the spectra of all $L_0(k_x)$.

Bloch Functions

Let $\{\lambda_s(k)\}_{s\in\mathbb{N}}$ and $\{\psi_s(k)\}_{s\in\mathbb{N}}$ be the eigenvalues and eigenfunctions of $L_0(k_x, k)$, i.e. $L_0(k_x, k)\psi_s(k) = \lambda_s(k)\psi_s(k)$. These are analytic functions in k on B.

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The Bloch functions are complete: for any $r \in L^2(\Omega)$ we have

$$r(\vec{x}) = \frac{1}{\sqrt{2\pi}} \sum_{s\in\mathbb{N}} \int_{-\pi}^{\pi} \langle \mathrm{U}\, r(\cdot, k), \psi_s(\cdot, k)\rangle \, \psi_s(\vec{x}, k)\; dk.$$

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We have the resolvent representation

$$\left((L_0(k_x) - \lambda)^{-1} r\right)(\vec{x}) = \frac{1}{\sqrt{2\pi}} \sum_{s\in\mathbb{N}} \int_{-\pi}^{\pi} \frac{\langle \mathrm{U}\, r(\cdot, k), \psi_s(\cdot, k)\rangle\, \psi_s(\vec{x}, k)}{\lambda_s(k) - \lambda}\; dk$$

for λ outside the spectrum of $L_0(k_x)$.

Waveguide

On $L^2(\mathbb{R}^2)$ consider $L_1 u = -\mathrm{div}\ \varepsilon_1^{-1}\nabla u$, where

- ε_1 periodic in the x-direction,
- $\varepsilon_0(x, y) = \varepsilon_1(x, y)$ for $y \notin (0, 1)$,
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Floquet transform in the x-direction gives family of problems

$$L_1(k_x)u := -\mathrm{div}\ \varepsilon_1^{-1} \nabla u \tag{2}$$

in $L^2(\Omega)$ satisfying qp-boundary conditions (1) with $k_x \in B$.

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Aim: Fix k_x and assume (Λ_0, Λ_1) is a spectral gap for $L_0(k_x)$. Introduce a line defect into the crystal and compare $\sigma(L_1(k_x))$ and $\sigma(L_0(k_x))$.

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Let $\lambda_M(k_x, k_0) = \Lambda_1$ and assume that $\lambda_M(k_x, k) \neq \Lambda_1$ for all k different from k_0. Analyticity implies $\lambda_M(k_x, k) \leq \Lambda_1 + \alpha|k - k_0|^2$.

The Bilinear Forms

Recall $\varepsilon_0, \varepsilon_1 \in L^\infty$ with a positive lower bound.

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$$H^1_{qp}(\Omega) := \{u \in H^1_{\text{loc}}(\mathbb{R}^2) : u(\vec{x} + (m,0)) = e^{ik_x m} u(\vec{x}), m \in \mathbb{Z}, \vec{x} \in \mathbb{R}^2\}.$$

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$$B_j[u, v] := \int_\Omega \left(\frac{1}{\varepsilon_j(\vec{x})} \nabla u \overline{\nabla v} + u \overline{v} \right) \, d\vec{x}, \text{ for } u, v \in H^1_{qp}(\Omega), j = 0, 1.$$

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Equip $H^1_{qp}(\Omega)$ with the norm coming from B_0.
Let $\mathfrak{L}_j + 1$ be the operator in $H^{-1}_{qp}(\Omega)$ associated with B_j and $\mathfrak{G}_j = (\mathfrak{L}_j + 1)^{-1}$.

Proposition

- *$\mathfrak{L}_0$ and $\mathfrak{G}_0$ are self-adjoint in $H^{-1}_{qp}(\Omega)$.*
- *$\sigma(L_0(k_x)) = \sigma(\mathfrak{L}_0)$ and $\sigma(L_1(k_x)) = \sigma(\mathfrak{L}_1)$.*
- *The essential spectra of $\mathfrak{G}_0$ and $\mathfrak{G}_1$ coincide.*

Approach: Birman-Schwinger Type Reformulation

$(L_1(k_x) - \lambda)u = 0$ with $\lambda \in (\Lambda_0, \Lambda_1)$ is equivalent to

$$(\mathfrak{G}_1 - \mu)u = 0 \text{ for } \mu = \frac{1}{\lambda + 1} \in \left(\frac{1}{\Lambda_1 + 1}, \frac{1}{\Lambda_0 + 1} \right).$$

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Moreover,

$$\begin{aligned}
(\mathfrak{G}_1 - \mu)u = 0 \quad &\Leftrightarrow \quad (\mathfrak{G}_0 - \mu)u + (\mathfrak{G}_1 - \mathfrak{G}_0)u = 0 \\
&\Leftrightarrow \quad (I - \mu{\mathfrak{G}_0}^{-1})u + ({\mathfrak{G}_0}^{-1}\mathfrak{G}_1 - I)u = 0 \\
&\Leftrightarrow \quad u + (I - \mu{\mathfrak{G}_0}^{-1})^{-1}({\mathfrak{G}_0}^{-1}\mathfrak{G}_1 - I)u = 0.
\end{aligned}$$

Note that $(I - \mu{\mathfrak{G}_0}^{-1})^{-1} = -(\lambda + 1)(\mathfrak{L}_0 - \lambda)^{-1}$.

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Lemma

Let $K := \mathfrak{G}_0{}^{-1}\mathfrak{G}_1 - I$. Then $K : H^{-1}_{qp}(\Omega) \to H^{-1}_{qp}(\Omega)$ is positive and symmetric and $\operatorname{Ran} K \subseteq H^{-1}_{cs}$, the space of distributions on Ω with compact support in the y-direction.

The Inner Product Space $\mathcal{K}$ and A_μ

Set $\mathcal{K} = \overline{\mathrm{Ran}\, K}$ and let $P : H^{-1}_{qp}(\Omega) \to \mathcal{K}$ be the orthogonal projection. On $\mathcal{K}$, we introduce a new inner product given by $\langle f, g \rangle_{\mathcal{K}} := \langle Kf, g \rangle_{H^{-1}_{qp}}$.

$$A_\mu := P(I - \mu \mathfrak{G}_{\mathfrak{o}}{}^{-1})^{-1} K : \mathcal{K} \to \mathcal{K}, \text{ for } \mu \in \left(\frac{1}{\Lambda_1 + 1}, \frac{1}{\Lambda_0 + 1} \right).$$

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Lemma

- $(\mathfrak{G}_1 - \mu)u = 0$ *has a non-trivial solution iff* -1 *is an eigenvalue of* A_μ.
- A_μ *is symmetric and compact in* $\mathcal{K}$.

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Lemma

- *$(\mathfrak{G}_1 - \mu)u = 0$ has a non-trivial solution iff -1 is an eigenvalue of A_μ.*
- *A_μ is symmetric and compact in $\mathcal{K}$.*

$$\kappa(\mu) := \min_{u \neq 0} \frac{\langle A_\mu u, u \rangle_{\mathcal{K}}}{\langle u, u \rangle_{\mathcal{K}}}.$$

Lemma

For μ in the spectral gap $\left((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1}\right)$ of $\mathfrak{G}_0$ we have that $\mu \mapsto \kappa(\mu)$ is continuous and increasing.

Expression for numerator

Let $u \in \mathcal{K}$, (v_n) in $L^2(\Omega)$ such that $v_n \to Ku \in H^{-1}_{qp}(\Omega)$. Then

$$\begin{aligned}
\langle A_\mu u, u\rangle_{\mathcal{K}} &= \langle KA_\mu u, u\rangle_{H^{-1}_{qp}} \\
&= \langle A_\mu u, Ku\rangle_{H^{-1}_{qp}} \\
&= \left\langle P(I - \mu \mathfrak{G}_o{}^{-1})^{-1} Ku, Ku\right\rangle_{H^{-1}_{qp}} \\
&= \left\langle (I - \mu \mathfrak{G}_o{}^{-1})^{-1} Ku, Ku\right\rangle_{H^{-1}_{qp}} \\
&= \lim_{n\to\infty} \left\langle (I - \mu \mathfrak{G}_o{}^{-1})^{-1} v_n, v_n\right\rangle_{H^{-1}_{qp}} \\
&= \lim_{n\to\infty} \left\langle \mathfrak{G}_o (I - \mu \mathfrak{G}_o{}^{-1})^{-1} v_n, v_n\right\rangle_{L^2} \\
&= \lim_{n\to\infty} \sum_s \int_{-\pi}^{\pi} \frac{\left|\langle Uv_n, \psi_s\rangle_{L^2}\right|^2}{(1 - \mu(\lambda_s + 1))\,(\lambda_s(k) + 1)} dk.
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&= \lim_{n\to\infty} \left\langle \mathfrak{G}_\mathfrak{o}(I - \mu {\mathfrak{G}_\mathfrak{o}}^{-1})^{-1} v_n, v_n\right\rangle_{L^2} \\
&= \lim_{n\to\infty} \sum_s \int_{-\pi}^{\pi} \frac{\left|\langle U v_n, \psi_s\rangle_{L^2}\right|^2}{\left(1 - \mu(\lambda_s + 1)\right)\left(\lambda_s(k) + 1\right)} dk.
\end{aligned}$$

On the one hand, for a suitable test function $u \in \mathcal{K}$ (such that $[(\mathfrak{L}_\mathfrak{o} - \mathfrak{L}_\mathfrak{1})\psi_M(\cdot, k_0)][\mathfrak{G}_\mathfrak{1} u] \neq 0$),

$$\langle A_\mu u, u\rangle_{\mathcal{K}} \to -\infty \text{ as } \mu \to 1/(\Lambda_1 + 1).$$

Lower Estimate for $\kappa(\mu)$

Recall: $\lambda_M(k_x^0, k_0) = \Lambda_1$.

$$
\begin{aligned}
\langle A_\mu u, u\rangle_{\mathcal{K}} &\geq \lim_{n\to\infty} \sum_{s\geq M} \int_{-\pi}^{\pi} \frac{|\langle U v_n, \psi_s\rangle_{L^2}|^2}{(\lambda_s(k)+1)\,(1-\mu(\lambda_s+1))} dk \\
&\geq \frac{1}{1-\mu(\Lambda_1+1)} \lim_{n\to\infty} \sum_{s\geq M} \int_{-\pi}^{\pi} \frac{1}{\lambda_s(k)+1} |\langle U v_n, \psi_s\rangle|^2 \, dk. \\
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&= \frac{1}{1-\mu(\Lambda_1+1)} \lim_{n\to\infty} \langle \mathfrak{G}_0 v_n, v_n\rangle_{L^2} \\
&= \frac{1}{1-\mu(\Lambda_1+1)} \lim_{n\to\infty} \|v_n\|^2_{H^{-1}} = \frac{1}{1-\mu(\Lambda_1+1)} \|Ku\|^2_{H^{-1}} \\
&\geq \frac{1}{1-\mu(\Lambda_1+1)} \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty \left\|\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0}\right\|_\infty \|u\|^2_{\mathcal{K}}.
\end{aligned}
$$

Result on Generation of Spectrum

Theorem

Assume that

$$\left\| \frac{\varepsilon_1}{\varepsilon_0} \right\|_\infty \left\| \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0} \right\|_\infty < \frac{\Lambda_1 - \Lambda_0}{(\Lambda_0 + 1)}.$$

Then weak localization takes place, i.e. the problem

$$-\nabla \cdot \varepsilon_1(\vec{x})^{-1} \nabla u^{(k_x^0)} = \lambda u^{(k_x^0)}, \quad \vec{x} \in \Omega = (0,1) \times \mathbb{R}$$

has a nontrivial k_x^0-quasiperiodic solution $u^{(k_x^0)} \in L^2(\Omega)$ for some $\Lambda_0 < \lambda < \Lambda_1$.

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Proof.

Combining the upper and lower estimates with the knowledge that $\kappa(\mu)$ is monotonically increasing in μ and continuous, we obtain our main result from the Intermediate Value Theorem. □

Further Results

Theorem

Let $\Sigma = \{(s,k) \in \mathbb{N} \times [-\pi,\pi] : \lambda_s(k) = \Lambda_1\}$*. Assume*

(i) $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2)$.

(ii) $\varepsilon_i \geq c_0 > 0$ *for some constant* c_0 *and* $i = 0, 1$.

(iii) *The perturbation is nonnegative, i.e.* $\varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0$.

(iv) *There exists a ball* D *such that* $\varepsilon_1 - \varepsilon_0 > 0$ *on* D.

(v) *The band functions* λ_s *are not constant as functions of* $k \in [-\pi,\pi]$.

(vi) *There are* $\alpha > 0$ *and* $\delta > 0$ *such that for all* $(\hat{s},\hat{k}) \in \Sigma$ *and* $k \in [-\pi,\pi]$ *satisfying* $|k-\hat{k}| \leq \delta$*, we have* $\lambda_{\hat{s}}(k) \geq \Lambda_1 + \alpha|k-\hat{k}|^2$.

Then $|\Sigma|$ *is finite.*

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(iii) *The perturbation is nonnegative, i.e.* $\varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0$.

(iv) *There exists a ball* D *such that* $\varepsilon_1 - \varepsilon_0 > 0$ *on* D.

(v) *The band functions* λ_s *are not constant as functions of* $k \in [-\pi,\pi]$.

(vi) *There are* $\alpha > 0$ *and* $\delta > 0$ *such that for all* $(\hat{s}, \hat{k}) \in \Sigma$ *and* $k \in [-\pi,\pi]$ *satisfying* $|k - \hat{k}| \leq \delta$, *we have* $\lambda_{\hat{s}}(k) \geq \Lambda_1 + \alpha|k - \hat{k}|^2$.

Then $|\Sigma|$ *is finite. Moreover, let* $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty > 0$ *be sufficiently small. Then the number of eigenvalues of* $\mathfrak{L}_1$ *in the gap* (Λ_0, Λ_1) *equals* $|\Sigma|$.

Further Results

Theorem

Let $\Sigma = \{(s,k) \in \mathbb{N} \times [-\pi,\pi] : \lambda_s(k) = \Lambda_1\}$. *Assume*

(i) $\varepsilon_0, \varepsilon_1 \in L^\infty(\mathbb{R}^2)$.

(ii) $\varepsilon_i \geq c_0 > 0$ *for some constant* c_0 *and* $i = 0, 1$.

(iii) *The perturbation is nonnegative, i.e.* $\varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0$.

(iv) *There exists a ball* D *such that* $\varepsilon_1 - \varepsilon_0 > 0$ *on* D.

(v) *The band functions* λ_s *are not constant as functions of* $k \in [-\pi,\pi]$.

(vi) *There are* $\alpha > 0$ *and* $\delta > 0$ *such that for all* $(\hat{s},\hat{k}) \in \Sigma$ *and* $k \in [-\pi,\pi]$ *satisfying* $|k - \hat{k}| \leq \delta$, *we have* $\lambda_{\hat{s}}(k) \geq \Lambda_1 + \alpha|k-\hat{k}|^2$.

Then $|\Sigma|$ *is finite. Moreover, let* $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty > 0$ *be sufficiently small. Then the number of eigenvalues of* $\mathfrak{L}_1$ *in the gap* (Λ_0, Λ_1) *equals* $|\Sigma|$.

Remark

For the case when all fields are constant in the z-direction, the results for TM- and TE-fields together imply the same results for the Maxwell system.

Thank you
for your attention!