Extremal lattices.

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The classification ofi the densest sphere packings in Euclidean space is a very old and difficult problem. In dimension 3 this was the famous Kepler conjecture, proved by Thomas Hales only a few years ago. The problem becomes much easier if one restricts to lattice sphere packings, where the centers of the spheres form a group. The density function has only finitly many local extrema on the $(n(n+1)/2-1)$-dimensional space of similarity classes of n-dimensional lattices and Korkine, Zolotareff and Voronoi (~ 1900) developed an algorithm to enumerate all of them. This has been done up to dimension 8. Dimension 24 is the only other dimension where one knows the densest lattice, due to the existence of the famous Leech lattice. The Leech lattice is one example of an extremal even unimodular lattice: The theta series of an even unimodular lattice of dimension n is a modular form of weight $n/2$ for the full modular group $\mathrm{PSL}_2(\mathbf{Z})$. It has already been observed by Siegel that the theory of modular forms allows to explicitly upperbound the density of an even unimodular lattice of dimension n:

$$\min(L) \leq \lfloor \frac{n}{24} \rfloor + 1.$$

Lattices achieving equality are called **extremal**. Of particular interest are extremal lattices and codes in the "jump dimensions" - the multiples of 24.

Number of extremal lattices L.

n	8	16	24	32	48	72	80	$\geq 163,264$
L	1	2	1	$\geq 10^7$	≥ 3	≥ 1	≥ 5	0

I will give a construction of the extremal even unimodular lattice Γ of dimension 72 I discovered in summer 2010. The existence of such a lattice was a longstanding open problem. The construction that allows to obtain the minimum by computer is similar to the one of the Leech lattice from E_8 and of the Golay code from the Hamming code (Turyn 1967). Γ can also be obtained as a tensor product of the Leech lattice (realised over the ring of integers R in the imaginary quadratic number field of discriminant -7) and the 3-dimensional Hermitian unimodular R-lattice of minimum 2, usually known as the Barnes lattice. This Hermitian tensor product construction shows that the automorphism group of Γ contains the absolutely irreducible rational matrix group $(\mathrm{SL}_2(25)\times\mathrm{PSL}_2(7)) : 2$.