Project 01: Random walk models on graphs and groups
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Research interests of W. Woess

The central topic of the research of W. Woess is "Random Walks on Infinite Graphs and Groups", which is also the title of the quite successful monograph [7]. Here, Random Walks are understood as Markov chains whose transition probabilities are adapted to an algebraic, geometric, resp. combinatorial structure of the underlying state space. The main theme is the interplay between probabilisitc, analytic and potential theoretic properties of those random processes and the structural properties of that state space. From the probabilistic viewpoint, the question is what impact the particular type of structure has on various aspects of the behaviour of the random walk, such as transience/recurrence, decay and asymptotic behaviour of transition probabilities, rate of escape, convergence to a boundary at infinity and harmonic functions. Vice versa, random walks may also be seen as a nice tool for classifying, or at least describing the structure of graphs, groups and related objects.

The work of W. Woess is not limited to those aspects that concentrate on the link between random walks and structure theory. One one side, there is also a body of more "pure" work on infinite graphs, group actions, and also formal languages (which entered the scene via the free group). This comprises past and current collaboration with T. Ceccherini-Silberstein. On another side, some recent and less recent work concerns locally compact groups and their actions in relation with the computation of norms of transition operators, and harmonic functions on certain spaces that arise as so–called horocyclic products: past and current collaboration with S. Brofferio, M. Salvatori, L. Saloff-Coste and A. Bendikov.

Woess' research is interdisciplinary between several Mathematical areas: Probability – Graph Theory – Geometric Group Theory – Discrete Geometry – Discrete Potential Theory – Harmonic Analysis and Spectral Theory.

PhD Research Projects

As a general practice, the research topics assigned to new PhD students try to take into account the previously achieved knowledge, capabilities and interests of the candidates – of course remaining within the range of expertise of Woess.

Here are brief descriptions of a few show-cases.

1. Random walks on infinite trees and random processes on their boundaries

Nearest neighbour random walks on trees are very well understood. An ample treatment can be found in Chapter 9 of Woess' textbook [8]. In recent work, Bendikov together with coauthors delopped a conceptually well accessible construction & analysis of random processes on ultra-metric spaces. These spaces are just boundaries (at infinity) of trees. Part

of the recent work by Bendikov, Grigor'yan, Pittet and Woess [2] concerns a duality between random walks on trees and those processes on their boundary. This topic merits further exploration in different directions.

- Deeper understanding of the probabilistic properties of the random processes on the trees' boundaries and the duality.
- Information theoretic aspects.
- Asymptotic evaluation of transition densities on specific classes of cases.
- Generalisation to construct processes on boundaries of hyperbolic graphs via random walks on the graphs.

2. Horocyclic and wreath products: group actions and random walks. This work goes back to two roots: one is a long paper of Cartwright, Kaimanovich and Woess [3] which studies random walks on the stabilizer of an end in a homogeneous tree in the spirit of products of random affine transformations. The other (itself twofold) is in a paper of Soardi and Woess [6] (one of the most frequently cited among Woess' papers), where amenability and unimodularity of totally disconnected groups are linked with random walks on graphs, and where Woess posed the following problem: does there exist a vertex-transitive graph that is not quasi-isometric with a Cayley graph ?

Diestel and Leader proposed an example to answer the above question positively. It can be described as the *horocyclic product* of two homogeneous trees with degrees $q+1$ and $r+1$, respectively where $q \neq r$ (that is, a horosphere in the product of two trees). It was only very recently that Eskin, Fisher and Whyte [4], [5] succeeded to prove that these *Diestel-Leader graphs* are indeed not quasi-isometric with any Cayley graph. On the other hand, as pointed out by R. I. Möller, when $q = r$, it *is* a Cayley-graph of the so-called lamplighter group, the wreath product of a finite group with the infinite cyclic group. All these facts opened the doors to work with multiple flavour.

First of all, many properties of random walks on DL-graphs could be studied in a series of papers by Woess and coauthors. Second, two classes of analogous structures were also studied in a similar spirit: the manifolds $\mathsf{Sol}(p, q)$ which are horocyclic products of two hyperbolic planes, and the treebolic spaces $\mathsf{HT}(p, q)$ which are the horocyclic products of a tree and a hyperbolic plane.

A survey and many references can be found in the article by Woess [9].

Bartholdi, Neuhauser and Woess [1] introduced horocyclic products of an arbitrary number of trees and studied them under several different viewpoints, including "pure" graph & (combinatorial) group theory besides spectral and boundary theory of random walks.

This is a starting point for further promising research to be undertaken. Orient the regular tree $T = T(d_1, d_2)$ such that every vertex has indegree d_1 and outdegree d_2, and consider the associated "horocycles" plus horocyclic products of two or more such trees, possibly with different d_i. From the above, the case $d_1 > d_2 = 1$ is well understood, where the involved groups are all amenable, which is not true when $d_1, d_2 \geq 2$.

When are those extended horocyclic products amenable, resp., when do they satisfy a strong isoperimetric ineqality ? This appears to be the simplest of the questions to be posed,

and a good starting point for a young PhD student to embark on this type of research by learning tools such as the action of vertex-stabilizers, the topology of isometry groups of graphs, related harmonic analysis, probability, etc.

What can be said about group actions on and quasi-isometries of these products ? How can one extend the methods of [1] to show that they are Cayley graphs when they are amenable ? Did such groups appear in Geometric Group Theory before in different contexts ? How are these graphs related with wreath products, Baumslag-Solitar and related groups ? What is the behaviour at infinity of such random walks ? Can one compute their spectra ?

Some related work has been done by the DK-student J. Cuno in the first DK-phase: it concerns random walks non-amenable Baumslag-Solitar groups, which are described via the horocyclic product of $T(d_1, d_2)$ with hyperbolic upper half plane.

These two showcases provide a selection, resp. flavour of possible PhD themes.

To summarize, we all know that it is a subtle task to find and propose good topics for PhD research in Pure Mathematics. On one hand, answers should not be obvious, while on the other, a responsible advisor should only propose questions which appear to have the potential for successful work within reasonable time. The above topics seem to have that potential, and to offer promising work to even more than one or two PhD students.

References

[1] Bartholdi, L., Neuhauser, M., and Woess, W.: *Horocyclic products of trees*, J. Eur. Math. Soc. (JEMS) **10** (2008) 771–816.

[2] Bendikov, A., Grigor'yan, A., Pittet, Ch., and Woess, W: *Isotropic Markov semigroups on ultra-metric spaces,* Uspekhi Mat. Nauk 69 (2014) No. 4 (418), 3-102 (in Russian). English original in Russian Math. Surveys 69 (2014) No. 4, 589-680.

[3] Cartwright, D. I., Kaimanovich, V. A., and Woess, W.: *Random walks on the affine group of local fields and of homogeneous trees,* Ann. Inst Fourier (Grenoble) **44** (1994) 1243–1288.

[4] Eskin, A., Fisher, D., and Whyte, K.: *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, Ann. of Math. **176** (2012) 221–260.

[5] Eskin, A., Fisher, D., and Whyte, K.: *Coarse differentiation of quasi-isometries II: Rigidity for Sol and lamplighter groups*, Ann. of Math. **177** (2013) 869–910.

[6] Soardi, P. M., and Woess, W.: *Amenability, unimodularity, and the spectral radius of random walks on infinite graphs*, Math. Z. **205** (1990) 471–486.

[7] Woess, W.: *Random Walks on Infinite Graphs and Groups*, Cambridge Tracts in Mathematics **138**, Cambridge University Press, Cambridge, 2000.

[8] Woess, W.: *Denumerable Markov chains. Generating functions, boundary theory, random walks on trees.* EMS Textbooks in Mathematics. European Mathematical Society, Zürich, 2009.

[9] Woess, W.: *What is a horocyclic product, and how is it related to lamplighters ?*, Internat. Math. Nachrichten of the Austrian Math. Soc. **224** (2013) 1-27.

Papers by Woess and coauthors can be found at

`http://www.math.tugraz.at/~woess/#papers`

(For opening the pdf files, please insert username Woess and password Vienna.)