1. Project 06: Polynomial diophantine equations - combinatorial and number theoretic aspects

1.1. Principal Investigator: Peter Kirschenhofer
Lehrstuhl für Mathematik und Statistik
Department Mathematik und Informationstechnologie
Montanuniversität Leoben
Franz Josef Str. 18
A-8700 Leoben
AUSTRIA

1.2. Keywords: diophantine equations, relative Thue equations, combinatorial enumeration problems, orthogonal polynomials, zero sets of linear recurrences

1.3. Research interests of the Faculty Member. The main research interest of the proposer is in the areas of Discrete Mathematics and algorithmic problems in Theoretical Computer Science, he has published altogether about 90 scientific papers within this scope. During the last years research was mainly focussed on problems in Combinatorics and Number Theory, in particular enumerative combinatorics and combinatorial identities (bijective and formal operator approaches), polynomial diophantine equations (polynomials originating from combinatorial enumeration problems, relative Thue equations, elements of small norm in extensions of imaginary-quadratic number fields, generalization of Waring’s problem), digital expansions and number systems (nonlinear recurrence relations in connection with SRS-systems). The proposer is interested in particular in problems that allow to combine techniques from two or more of the referred areas of Discrete Mathematics. Research has been performed within several projects granted by the FWF as principal investigator or co-investigator, three of them within FWF-National Research Networks. In particular the proposer is co-applicant of a project within the actual FWF-NFN S96 on "Analytic Combinatorics and Probabilistic Methods in Number Theory", which means that research within the current proposal could be carried out in interaction with this Austrian network. The proposer headed also a joint project with the University of Debrecen, Hungary, funded by the "Austrian Academic Exchange Service" within the last years. Two PhD theses in thematic relation to this proposal have been successfully completed under the guidance of the proposer within the last five years.

1.4. Short description of two showcases of PhD Research Projects. Combinatorial polynomial diophantine equations:
Polynomial diophantine equations have attracted special interest during the past decades. Whereas the number theoretic point of view constitutes the main focus of research in this area one further important source of problems originates from combinatorics: there is a variety of combinatorial enumeration problems leading to counting polynomials where certain recurrence relations reflect the original combinatorial structure. Enumerative questions correspond to the investigation of the number of solutions of diophantine equations between the concerned polynomials then. Several important families of orthogonal polynomials resp. so-called Sheffer polynomials appear within this combinatorial schemes.
For example let $p_n(k)$ denote the number of random paths in the positive integer grid, consisting of horizontal, vertical or diagonal steps (leading to the right resp. upwards), and connecting the origin with the point $(n,k)$. (The $p_n(k)$ are sometimes attributed as the Delannoy numbers). Using direct combinatorial arguments resp. generating functions it can be shown that the
polynomials \( q_n(t) := i^n n! p_n(-\frac{1}{2} - i \frac{t}{2}) \) fulfill the second order recurrence

\[
q_{n+1}(t) = t q_n(t) - n^2 q_{n-1}(t), q_{-1}(t) = 0, q_0(t) = 1.
\]

Consequently the \( q_n(t) \) are real orthogonal polynomials; in fact they form a special instance of the Meixner-Pollaczek polynomials

\[
q_n(x) = n! P_{n}^{(\frac{1}{2})}(\frac{x}{2}, \frac{\pi}{2})
\]

where \( P_{n}^{(\lambda)}(x, \phi) \) denotes the Meixner-Pollaczek polynomials

\[
P_{n}^{(\lambda)}(x, \phi) = \frac{(2\lambda)_n e^{i\phi}}{n!} 2F_1 \left[ -n, \lambda + ix \frac{2}{2\lambda}; 1 - e^{-2i\phi} \right]
\]

The polynomial diophantine equations

\[
p_m(x) = p_n(y)
\]

have been extensively studied by several authors (compare e.g. \cite{7}) making use of their orthogonality; we refer also to the results by Stoll and Tichy \cite{13} in this context. In his PhD thesis O.Pfeiffer (directed by the principal investigator of this project) has established results for polynomial diophantine equations originating from a general combinatorial scheme yielding recurrence equations of the form

\[
p_{n+1}(x) = xp_n(x) + c_n p_{n-1}(x)
\]

and established conditions on the parameters \( c_n \) that guarantee finiteness of the number of solutions of the corresponding diophantine equations \( p_m(x) = p_n(y) \), compare also \cite{9}.

In the present project we intend to continue research on combinatorial polynomial diophantine equations along the following lines:

- The identification of more general classes of recurrence relations that result in finiteness of solutions of the corresponding polynomial diophantine equations (including a possible identification of the solutions themselves). From the methodological point of view research will have to include a thorough investigation of properties of the polynomials in the families in consideration. Considering for instance a well-known result by Bilu and Tichy \cite{3} one issue of particular importance in this context will be the problem of multiplicities of extremal values: whereas for orthogonal polynomials obeying differential equations this problem is settled by Sonin’s theorem corresponding results for the members of orthogonal families obeying difference equations do not seem to be known and should be derived within this project.
- The identification of combinatorial structures corresponding to the recurring sequences according to the last item. Here we seek on the one hand for direct resp. "bijective" combinatorial proofs for the recurrences in question in order to get sufficient insight into the combinatorial structures behind, on the other hand formal operator methods will also allow to establish generating functions yielding the desired recurrence relations. Combinatorial approaches of this type have been followed, for example, in \cite{8} or \cite{9} and should be continued within the present project.
- A combination of results according to the last two items could constitute the basis for finding more general contexts between the (discrete) structure of combinatorial objects and the finiteness of the number of solutions of corresponding polynomial diophantine problems. We would aim to come one step further towards a kind of "automatized"
analysis of diophantine behavior of combinatorial structures in the sense discussed above.

**Parameterized families of relative Thue equations:**
Another important development is in concern of polynomial diophantine equations over number fields. Various results could be achieved in the past years with respect to parameterized families of relative Thue equations over imaginary quadratic fields. For example let $k = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic number field with ring of integers $\mathbb{Z}_k$ and consider the family of relative Thue equations

$$F_t(x, y) = x^3 - (t - 1)x^2y - (t + 2)xy^2 - y^3 = l$$

with $t, l \in \mathbb{Z}_k$, $t \notin \mathbb{Z}$ and $|l| \leq |2t + 1|$. We are interested in the solutions of these "diophantine" equations, i.e. in the pairs $(x, y) \in \mathbb{Z}_k \times \mathbb{Z}_k$ satisfying the equation. The corresponding equations in the rational integer case were solved by Mignotte et al. in [12], whereas the instance $|l| = 1$ of the relative Thue equations was considered by Heuberger et al. in [5] and [4]. One of the main tools in treating the above Thue equation turns out to be the following. Let $k(\alpha)$ be the cubic extension of $k$ generated by the polynomial

$$f_t(x) = x^3 - (t - 1)x^2 - (t + 2)x - 1$$

and let $(x, y) \in \mathbb{Z}_k \times \mathbb{Z}_k$ be a solution of the relative Thue equation. Denoting by $N_{k(\alpha)/k}(\gamma)$ the relative norm of $\gamma$ over $k$ we have

$$N_{k(\alpha)/k}(x - \alpha y) = F_t(x, y) = l.$$ 

Therefore, solving the Thue equation for $|l| \leq |2t + 1|$ is equivalent to determining all elements $\gamma = x - \alpha y$ whose norm is bounded by $|2t + 1|$ in absolute value. Following this idea in a first step in [10] all elements $\gamma \in \mathbb{Z}_k[\alpha]$ with relative norms satisfying $|N_{k(\alpha)/k}| \leq |2t + 1|$ were investigated for large $|l|$. This generalizes a corresponding result by Lemmermeyer and Pethö [11] for Shanks’ cubic fields over the rationals. In a PhD thesis by C. Lampl advised by the principal investigator and J. Thuswaldner this result was completed for all values of $t$ and applied in order to solve the relative Thue equation for all $t$ with $\Re t = -\frac{1}{2}$ (compare [6]).

In the present PhD project we would aim to continue research within the following directions.

- Extension of the above cited results to the instance $\Re t \neq -\frac{1}{2}$. In this general instance we will have to consider linear forms in one additional logarithm compared with the situation from before, which means that refinements of Baker’s theorems on linear forms in logarithms like Mignotte’s results in [12] will have to be adjusted. Of course we would aim at deriving results for other parameterized families of relative Thue equations, too.

- Special instances of the above problem where Baker’s method does not yield solution free regions lead to the question of the multiplicity of zeros and properties of the zeros of linear recurrence relations in the integers of the referred number fields. With respect to the relative Thue equations mentioned above for $\Re t = -\frac{1}{2}$ the following ternary recurrence has to be investigated.

$$p_m + 3p_{m-1} + Lp_{m-2} + p_{m-3} = 0 \quad (m \geq 3),$$

with $L := -(t^2 + t + 4) \in \mathbb{Z}$, $p_0 = 0$, $p_1 \in \{0, 1\}$ and $p_2 = p_2(c) \in \mathbb{Z}$ where $c$ is defined by $t = -\frac{1 + \sqrt{\Delta}}{2}$. The solution is a sequence of polynomials

$$p_m = p_m(L, c)$$

with coefficients in $\mathbb{Z}$ and for the purposes of the solution of the relative Thue equation we are interested in all instances where $p_m(L, c) = 0$. For $\Re t \neq \frac{1}{2}$ complex recurrences
will have to be treated. This is in context with a classical problem; the structure of the set of values of the recurrence is treated by the Skolem-Mahler-Lech theorem, particular results are due to Beukers, Schlickewei, W.M.Schmidt, Tijdeman and others; special results on the zero set of binary complex recurrences were established by Beukers and Tijdeman [2], or for ternary recurrences within the rational numbers by Beukers [1]. A generalization of the latter results to recurrences of higher degree would be an important tool for the treatment of the relative Thue equations in question. In those instances where the zero set cannot be determined explicitly, special information could also be drawn by congruence methods on divisibility properties of the indices of the zero elements of the recurrence.

1.5. Collaborations within the DK-plus. The present project will allow cooperations with several other projects within the doctoral program. There will be close connections to problems studied within Project 09 by R. F. Tichy, in particular with respect to combinatorial problems underlying the diophantine problems studied there. Further connections will occur with Project 05 by C. Heuberger with whom we share common interest in Diophantine equations and with Project 08 by J. Thuswaldner with respect to diophantine problems and number systems.

1.6. Collaborating research groups where PhD Students could perform their research stay abroad.

- Dominique Foata, Institute Lothaire, Universite Louis Pasteur, Strasbourg
- Peter J. Larcombe, Derby Business School, University of Derby, England
- Attila Pethő, Department of Informatics, University of Debrecen, Hungary.

1.7. Know-how and infrastructure of the research group. The proposer is Full Professor at the Chair for Mathematics and Statistics of the Department Mathematics and Information Technology at the University of Leoben.

Apart from the two scientists that are applying for a project within the current doctoral program (P. Kirschenhofer and J. Thuswaldner) there are several other scientists working on Discrete Mathematics. One of them is W. Imrich (he will retire this year) who works on graph theory. Moreover, we have two post docs and one PhD student who are employed within the currently running FWF national research network S96. They work on topics related to Projects 06 and 08 of the proposed program.

We emphasize that several other projects funded by the FWF have been carried out successfully (resp. are currently carried out) at the department in recent years. We mention the research projects FWF-P14200, FWF-P17557, FWF-P18969 and FWF-20989, project FWF-S8310, which was part of the research network FWF-FSP-S83 as well as Projects S6910 and (partially) S9611 which belong to the currently running national research network FWF-NFN-S96.

Moreover, at the department working space and office room as well as computer equipment is available in order to supply a good working environment for the PhD students to be employed on the two positions we apply for within the current program (note that the proposer of Projects 08 is working at the above-mentioned department; he has also applied for one PhD position). Moreover, a well equipped mathematical library is present at the department. Since the proposer as well as several of his colleagues at the department have many scientific contacts inside Austria and abroad, there is a vivid scientific exchange taking place at the department.
1.8. CV of the Faculty Member.

Education:

- 5.7.1956: born in Vienna
- 1974–1975: Military service
- 1975–1979: Studies of Mathematics, Physics and Chemistry at the University of Vienna
- 1979: PhD in Mathematics, University of Vienna, thesis advisor J.Cigler
- 1983: Habilitation (venia docendi) for “Discrete Mathematics”, Vienna University of Technology

Employment and visiting positions.

- 1979–1996: Assistant Professor at the Department of Applied Mathematics, later on at the Department of Algebra and Discrete Mathematics, Technical University of Vienna
- 1989–1991: Visiting Professor at the University of Klagenfurt
- 1996–: Full Professor of Mathematics and Mathematical Statistics, University of Leoben, Austria
- 1997–1999: Department Head, Department of Mathematics and Applied Geometry, University of Leoben
- 2001–: Chairman of the Senate, University of Leoben
- 2007–: Department Head, Department of Mathematics and Information Technology, University of Leoben

Organization of conferences:

- 1984–: Co-organizer of several international conferences and seminars on Combinatorics and Theoretical Computer Science
- September 2007: Member of program committee of Austrian-Slovak Joint Conference of Mathematical Societies in Podbanske, Slovakia

Awards:

- 1989: Scientific Prize of the Austrian Mathematical Society

Scientific projects:

- Several international cooperation projects, in particular with Ph.Flajolet (INRIA, France), C.Martinez (Universidad Politecnica Barcelona), A.Pethő (Debrecen, Hungary), W. Szpankowski (Purdue University, West Lafayette, USA)
- FWF-NSF joint science project P7497-TEC “Analysis of data structures for digital search” (principal investigator) together with H.Prodinger, and W.Szpankowski (Purdue University, West Lafayette, USA),
- FWF project P14200-N05 “Toolkits for the Average Case Analysis of Algorithms” (principal investigator)
- FWF project S8307-MAT “Algorithmic Diophantine Equations” (co-investigator, principal investigator R.F.Tichy, TU Graz)
- FWF project S8310-N12 “Combinatorial Analysis of Number Theoretic Algorithms” (principal investigator)
Referee for scientific organisations:

- referee for a variety of international journals in the fields of Discrete Mathematics and Theoretical Computer Science
- referee for "Austrian Academic Exchange Service" (international scientific exchange projects)
- referee for several appointment committees, "habilitation" committees and doctoral committees at other universities

Supervision of PhD theses:

- 1985, TU Vienna, J. Blieberger, "Asymptotic investigations on the number of generalized Motzkin trees and related problems" (in German); J.Blieberger is now Associate Professor at the Technical University of Vienna
- 1992, TU Vienna, W.Schachinger, "Contributions to the analysis of data structures for digital search" (in German); W.Schachinger is now Associate Professor at the University of Vienna
- 1993, TU Vienna, F.Hubalek, "Contributions to the analysis of generalized digital search trees" (in German), F.Hubalek is now Assistant Professor at the Technical University of Vienna
- 2003, MU Leoben, O.Pfeiffer, "Combinatorial analysis of problems in algorithmic number theory: finiteness theorems for polynomial diophantine equations" (in German)
- 2007, MU Leoben, C.Lampl, "Families of relative Thue equations over imaginary quadratic number fields" (in German)

1.9. PhD Students of the last 5 years.

<table>
<thead>
<tr>
<th>Name of the student</th>
<th>Sex</th>
<th>Research topic</th>
<th>Title of the thesis</th>
<th>Number of publications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Catrin Lampl</td>
<td>F</td>
<td>diophantine equations</td>
<td>Families of relative Thue equations over imaginary quadratic number fields (in German)</td>
<td>1</td>
</tr>
<tr>
<td>Oliver Pfeiffer</td>
<td>M</td>
<td>diophantine equations</td>
<td>Combinatorial analysis of problems in algorithmic number theory: finiteness theorems for polynomial diophantine equations (in German)</td>
<td>2</td>
</tr>
</tbody>
</table>

1.10. Externally funded national and international projects (last 5 years).

<table>
<thead>
<tr>
<th>Funding organization</th>
<th>Number of the project</th>
<th>Research topic of the project</th>
<th>Amount funded in KEURO</th>
</tr>
</thead>
<tbody>
<tr>
<td>FWF</td>
<td>P14200-N05</td>
<td>Toolkits for the Average Case Analysis of Algorithms</td>
<td>57,5</td>
</tr>
<tr>
<td>FWF</td>
<td>S8310-N12</td>
<td>Combinatorial Analysis of Numbertheoretical Algorithms</td>
<td>146,5</td>
</tr>
</tbody>
</table>
1.11. Most relevant papers of the last 5 years.


References of Project 06


