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Short description of three showcases of PhD Research Projects

Showcase 1. Multidimensional random walks with reflections

Reflected random walk (RRW) on $\mathbb{Z}^+$ (or $\mathbb{R}^+$) starts like an ordinary random walk as a sum of i.i.d. random variables $(Y_n)_{n\geq 1}$, but when it becomes negative, the sign is reversed before the process continues. Thus, the process starting at $x \geq 0$ is given recursively by $X_0^x = x$ and $X_n = |X_{n-1}^x - Y_n|$. RRW goes back to considerations of telephone networks; in the probability literature it is present since Feller [1971] and has been studied up to today; see e.g. Peigné and Woess [2011].

Woess former Phase 2 DK student Kloas has been working on the multi-dimensional variant of RRW on $\mathbb{Z}^d$ (or $\mathbb{R}^d$), see Kloas and Woess [2018]. It is defined analogously, with reflection as above in some or all coordinates. There are applicatons, for example, to queueing theory, see Cygan and Kloas [2018]. The multi-dimensional case is harder than dimension 1, and these two papers are the starting point for further work to be done. *The asymptotics of transition probabilities* are not only a possible tool for understanding recurrence/transience; they are also of great interest in themselves. So far, only few results are available even for RRW in dimension 1, see Essifi and Peigné [2015]. We shall investigate possible extensions of their methods. Also, the in the transient case, the boundary behaviour of multidimensional RRW is of great interest.

At first glance, RRW seems to be similar to the fascinating topic of *lattice random walks in cones.* So far, however, the approaches and methods have been very different. While for RRW, one is interested in unbounded jumps, here one restricts a – typically nearest neighbour – random walk on a lattice structure in $\mathbb{R}^d$ to the positive cone $\mathbb{R}^d_+$ and imposes specific transition rules at the boundary. There is a large body of work on this subject, see the monograph by Fayolle et al. [1999] and a lot of impressive recent work on lattice walks in the quarter plane, where the way of thinking is to a large extent that of subtle combinatorial counting arguments. The exploration of meeting points between the two theories is also a promising field.

This research is particularly well suited for international collaboration with the research group at Tours (France), in particular March Peigné and Kilian Raschel, who have a close exchange with Woess' group at Graz.

Showcase 2. External DLA on the Sierpinski gasket (in collaboration with Ecaterina Sava-Huss)

External DLA was initially introduced in physics by Witten and Sander [1983], as an example to create ordering out of chaos due to a simple rule. Mathematically, this ordering is far away from being understood. External DLA on an infinite state space G is a model of random fractal growth which builds a sequence of randomly growing sets $(\mathcal{E}_n)_{n\geq 0}$, starting with one particle $\mathcal{E}_0 = \{o\}$ at the origin of G. At each time step, a new particle starts a simple random walk from "infinity" (far away) and walks until it hits the outer boundary of the existing cluster, where it stops and settles. In this way, one builds a family $(\mathcal{E}_n)_{n\geq 0}$ of growing clusters; the set $\mathcal{E}_n$ consists of exactly $n+1$ particles and it is called *external DLA cluster.* In spite of these very simple growth rules, only a few rigorous mathematical results about external DLA are available. A typical structure produced on a two-dimensional lattice is shown in Figure 1. External DLA was found to well represent growth processes in nature

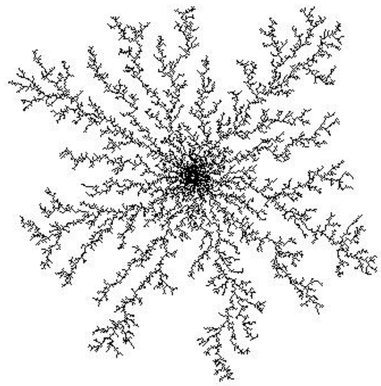

FIGURE 1. External DLA cluster with center initially occupied

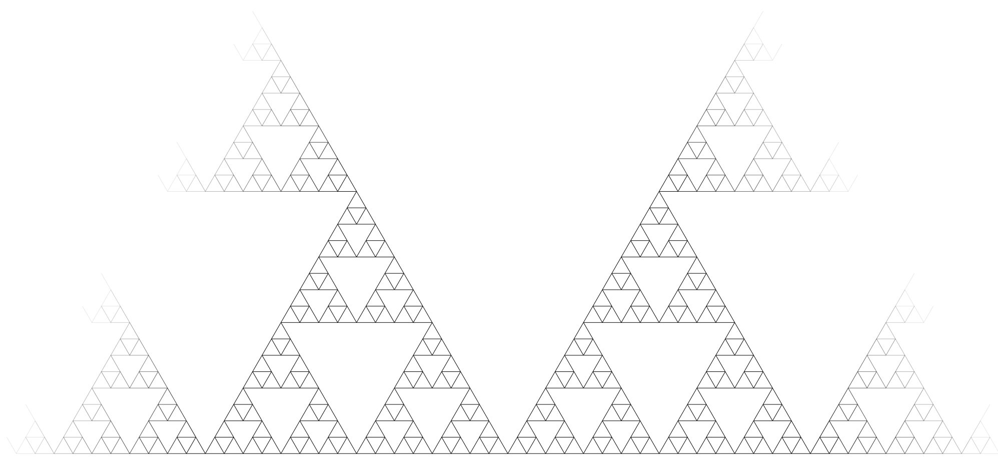

FIGURE 2. Doubly-infinite Sierpinski gasket graph SG.

such as growth of bacterial colonies, electrodeposition, crystal growth. An overview of existing results on external DLA is given in Sava-Huss [2009].

We propose as a possible research topic for a PhD student, the investigation of external DLA model on the Sierpinski gasket graph SG. See Figure 2 for a graphical representation of SG. External DLA on SG seems to be an approachable problem, due to the fact that SG is a p.c.f. fractal, and the existence of cut points may simplify the understanding of the harmonic measure from infinity. Possible questions one could investigate are listed below. Computer simulations may also help in understanding the behavior of the proposed model on the Sierpinski gasket graph.

Question. Can one find an upper bound for the growth of arms in external DLA on SG by extending the method of Kesten [1987], Kesten [1990]?

Question. Does the external DLA cluster on the Sierpinski gasket graph have infinitely many holes, with probability one, as in the case of $\mathbb{Z}^2$ from Eberz-Wagner [1999]?

Once we understand how this model behaves on the gasket, we can look at other finitely ramified fractal graphs, and ask the same two questions.

Showcase 3. The language of self-avoiding walks

This showcase concerns research that was begun a few months ago by Woess' new PhD student *Christian Lindorfer*, who has been designated as an associated DK student who will be present for about 3 years during the forthcoming 3rd DK funding period. That is, this is an already activated showcase for Phase 3.

Consider a locally finite graph (X, E) without loops. Let σ_n be the number of self-avoiding walks (i.e., walks that do not self-intersect) starting at a root vertex o . The *connective constant* of the graph is $\sigma = \limsup \sigma_n^{1/n}$. While in itself of combinatorial nature, this object has been studied intensively in environments of Statistical Physics and Probability. See the monograph by Madras and Slade [2013] and the lecture notes of Bauerschmidt et al. [2012]. It is in general hard to determine σ explicitly.

A *labelled graph* is obtained by replacing each edge by two oppositely oriented edges. Each oriented edge e then gets a label $\ell(e)$ from a finite alphabet $\mathbf{\Sigma}$. We assume that the labelling is *deterministic:* at any vertex, for any label $a \in \mathbf{\Sigma}$, there is at most one outgoing edge at x with label a. Typical examples are Cayley graphs of finitely generated groups: such a group is a factor of a free group, whose free generators + inverses form the alphabet $\mathbf{\Sigma}$. See e.g Ceccherini-Silberstein and Woess [2012] for details in this spirit.

For any finite walk in the graph, the successive labels along its edges form a word over $\mathbf{\Sigma}$. Then the language of self-avoiding walks is the language of all words that are formed by reading the labels of some self-avoiding walk starting at o. What is the nature of this language within the Chomsky hierarchy of formal languages and some later extensions ?

For comparison, consider the language of all *closed* walks from o back to o (possibly self-intersecting). In important work of Muller and Schupp [1985], the class of *context-free graphs* was introduced: in brief, these are precisely the labelled graphs where the language of all words along *closed* walks is context-free. For a Cayley graph of a group, this means that the group is virtually free, a famous result of Muller and Schupp [1983]. The application to random walks was outlined by Woess [1987].

The analogous question regarding self-avoiding walks was inspired by a note of Zeilberger [1996] on "ladder graphs" and another, recent note of Gilch and Müller [2017] on free products of finite graphs. If the language of self-avoiding walks is regular, resp. context free, then this yields that the generating function $\sum_n \sigma_n z^n$ is rational, resp. algebraic, so that the connective constant – the inverse of its radius of convergence – is an algebraic number.

Indeed, in famous work, Duminil-Copin and Smirnov [2012] have even proved that the connective constant is an algebraic number, namely $\sqrt{2+\sqrt{2}}$, for the hexagonal lattice, which is not a context-free graph.

The initial work of Lindorfer has lead to the following two conjectures: a) for a quasi-transitive labelled graph, the language of self-avoiding walks is regular if and only if all ends of the graph have size 1. b) The language is context-free if and only if all ends of the graph have size (the number of independent infinite rays representing an end) at most 2; compare with the terminology of Thomassen and Woess [1993]. For groups, this can be formulated in terms of free products and HNN-extensions over subgroups with cardinality 1, resp. 2.

Subseqent questions concern the relation with further classes of formal languages, e.g. those studied by Ceccherini-Silberstein et al. [2015]. A language-theoretic approach to self-avoiding walks appears completely new and promising, and may open a link to word processing in groups, referring to the famous book by Epstein et al. [1992] with that title.

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