Faculty member Alfred Geroldinger (Project 03)

Short description of two showcases of PhD Research Projects

Showcase 1. Sets of Lengths in Krull monoids.
Let H be a Krull monoid with finite class group G such that every class contains a prime divisor (principal orders in algebraic number fields are classical examples of such Krull monoids). Then every $a \in H$ has a factorization into irreducibles, say $a = u_1 \ldots u_k$. In this case k is called a factorization length of a and $\mathsf{L}(a)$ denotes the (finite) set of all possible factorization lengths. We consider the whole family $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$. Sets of lengths in H depend just on the class group G. Indeed, for the monoid $\mathcal{B}(G)$ of zero-sum sequences over G we have $\mathcal{L}(H) = \mathcal{L}\big(\mathcal{B}(G)\big)$. A side remark: this well-known transfer result from H to $\mathcal{B}(G)$ was extended by D. Smertnig (a former DK student in the project of Geroldinger) from Krull monoids to large classes of non-commutative Dedekind domains including maximal orders in central simple algebras over number fields (Baeth and Smertnig [2015]; Smertnig [2013]).

Sets of lengths in $\mathcal{B}(G)$ are studied with methods from additive combinatorics. It is well-known that all sets of lengths are singletons (i.e., $|L| = 1$ for all $L \in \mathcal{L}(B(G))$) if and only if $|G| \le 2$ and if $|G| \ge 3$, then for every $n \in \mathbb{N}$ there is a set of lengths of size n. Sets of lengths have a well-defined structure. They are almost arithmetical multiprogressions with a global bound on all parameters ([Geroldinger and Halter-Koch 2006, Chapter 4]), and this description is best possible (Schmid [2009]).

The parameters controlling these multiprogressions are arithmetical zero-sum invariants which are of a similar nature as the classical combinatorial zero-sum invariants, such as the Davenport constant $\mathsf{D}(G)$ and the Erdős-Ginzburg-Ziv constant of G. Some arithmetical invariants are closely connected to the combinatorial invariants (e.g., the elasticity of $\mathcal{B}(G)$ equals $\mathsf{D}(G)/2$). There is strong recent progress on the classical combinatorial constants by Girard, Pach, Plagne, Schmid, et al. (Croot et al. [2017]; Girard [2018]; Plagne and Schmid [2019]) and this offers new starting points for studying the (minimal) set of distances and the elasticities (the state of the art is presented in the survey Schmid [2016]).

The long term goal in all these investigations on arithmetical invariants is the Characterization Problem. As outlined above, sets of lengths in H depend just on the class group G, and the Characterization Problem is an associated inverse problem. It asks whether sets of lengths are characteristic for the class group. The problem runs as follows.

> *Let G be a finite abelian group with $\mathsf{D}(G) \ge 4$ and let G' be any abelian group such that $\mathcal{L}\big(\mathcal{B}(G)\big) = \mathcal{L}\big(\mathcal{B}(G')\big)$. Are G and G' isomorphic?*

The standing conjecture is that the answer is always affirmative. Although there was strong progress in the last years (Zhong [2018a,b]; Geroldinger and Schmid [2019]), the problem is wide open.

The research in this project is well-suited for collaboration with colleagues in Paris (in particular, E. Balandraud, B. Girard, A. Plagne, and W. Schmid).

Showcase 2. Arithmetic of C-monoids.
C-monoids fulfill a variety of algebraic finiteness properties. If H is a C-monoid, then H satisfies the ascending chain condition on divisorial ideals, has a nonempty conductor $(H \colon \widehat{H})$ to its complete integral closure $\widehat{H}$, and $\widehat{H}$ is Krull with finite class group. Non-principal

orders in algebraic number fields are C-monoids and the same is true for the monoid $\mathcal{B}(G)$ of product-one sequences over a finite, not necessarily abelian group G.

Similarly to the case of Krull monoids, the arithmetic of a general C-monoid H can be studied in an associated monoid H_0 which allows to apply combinatorial methods (indeed, H_0 is a (suitable) submonoid of a finitely generated factorial monoid which inherits the above mentioned properties). These algebraic finiteness properties allow to derive abstract arithmetical finiteness properties. As in the Krull case, sets of lengths are again almost arithmetical multiprogressions and (minimal) sets of distances are finite. However, in most cases there are just abstract finiteness results but there are no reasonable upper bounds, let alone precise values for the arithmetical invariants. To tackle these problems we have the following two strategies in mind.

1. A detailed study of class groups and class semigroups. For simplicity, consider a non-principal order $\mathcal{O}$ in an algebraic number field. Then the arithmetic of $\mathcal{O}$ does not only depend on the Picard group $\mathrm{Pic}(\mathcal{O})$ (as it is the case for principal orders) but also on the structure of the localizations at primes containing the conductor. Being a C-monoid the class semigroup of $\mathcal{O}$ is finite but it is not clear yet how the structure of the class semigroup influences the arithmetical invariants. First steps are done in case of the monoid of product-one sequences by JunSeok Oh, the present DK student in the project of Geroldinger (Oh [2018, 2019]).

2. The seminormal case. Seminormal C-domains are seminormal Mori domains. The impact of the seminormality property on the ideal theory of Mori domains has been studied since the 1990s (see the survey Barucci [1994]; seminormal orders in algebraic number fields were first studied by Dobbs and Fontana [1987]). In the last couple of years first steps were done in using these results on the ideal theory to obtain more precise results on the arithmetic (Geroldinger et al. [2015]; Geroldinger and Zhong [2016]). Among others, we could show that 1 lies in the set of distances of every seminormal order (in plain words, there are $k \in \mathbb{N}$ and irreducible elements $u_1, \ldots, u_k, v_1, \ldots, v_{k+1}$ such that $u_1 \ldots u_k = v_1 \ldots v_{k+1}$). This is trivially true in principal orders and still open for orders in algebraic number fields in general. JunSeok Oh could characterize the seminormality property of $\mathcal{B}(G)$ by structural properties of the class semigroup. We consider this as an initial result relating algebraic and arithmetic properties of a C-monoid with structural properties of its class semigroup.

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