Faculty member Peter Grabner (Project 04)

Short description of two showcases of PhD Research Projects

Showcase 1. Distribution of points on the unit sphere
Approximating uniform uniform distribution by finite point configurations on the sphere and other (homogeneous) manifolds is a question that has attracted considerable interest in recent years. The now classic survey Saff and Kuijlaars [1997] has been continued and extended recently in Brauchart and Grabner [2015]. Finding explicit point sets on $\mathbb{S}^2$, which have minimal order $N^{-3/4+\epsilon}$ for the discrepancy is a difficult task, even having at hand a number of constructions of good candidates such as spiral points Saff and Kuijlaars [1997] or Fibonacci points Aistleitner et al. [2012].

Projecting the points to the unit square by the classical Lambert-transformation transforms the problem into a planar counting problem for which exponential sum techniques as described in Krätzel [1988]; Graham and Kolesnik [1991] can be applied. A different approach would be to work out the harmonic analysis directly on the sphere using spherical harmonics. Depending on the construction of the point set, both methods could yield bounds for the discrepancy. Any bound for the discrepancy of a sequence of point sets better than $N^{-1/2}$, the probabilistic bound, would be an immense progress.

Of course, also higher dimensional generalisations would be of interest, but the problem seems to be already hard on $\mathbb{S}^2$.

Based on Torquato and Stillinger [2003] the notion of hyperuniformity has been generalised and extended to the compact setting, such as the sphere $\mathbb{S}^d$ Brauchart et al. [2018]. Hyperuniformity is described in terms of the fluctuations of the counting function

$$\#(X_N \cap C(\mathbf{x}, \phi)),$$

where $C(\mathbf{x}, \phi)$ is a spherical cap of angle ϕ around $\mathbf{x}$. A sequence of sets $(X_N)_N$ is called hyperuniform, if the variance of this counting function is of smaller order than for i.i.d. random points. The case of small caps, where $\phi \to 0$ with $N \to \infty$ is of interest, especially $\phi = tN^{-1/d}$. Again, explicit constructions of such point sets are still unknown, of the above mentioned constructions are candidates.

There is an intensive cooperation with the group of D. Hardin and E. Saff at Vanderbilt University, which would also allow for an exchange of doctoral students working on the subject.

Showcase 2. Determinantal point processes
Determinantal point processes have their motivation in physics, where they occur through the wave function of fermionic particle systems Hough et al. [2009]. In the context of point distributions on manifolds such as the sphere they are used to obtain random point configurations, which exhibit more regularity than i.i.d. random points by the built in mutual repulsion.

Several processes have been defined on spheres (see Beltrán and Etayo [2018]; Beltrán et al. [2016]; Alishahi and Zamani [2015]), which have been analysed with respect to their energies and discrepancies. The worst error in integration for different reproducing kernel Hilbert spaces still has to be investigated for these processes. Furthermore, similar processes can be defined on other homogeneous manifolds; the energy, discrepancy, and numerical integration errors can be investigated for them. Some work in this direction has already been done for flat tori Marzo and Ortega-Cerdà [2018], but there is a large number of open questions for a PhD student to work on.

Such point processes also provide a method to derive new upper bounds for the minimum of energy expressions

$$E_g(X_N) = \sum_{i\neq j=1}^{N} g(\|x_i - x_j\|),$$

where g is monotonically decreasing function of the distance with a possible singularity at 0, for example the Riesz function $\|x-y\|^{-s}$ for $s>0$. The upper bounds obtained from the expected value of the energy can be compared to lower bounds obtained by linear programming methods as developed in Boyvalenkov et al. [2016].

A new cooperation with the University of Cantabria in Santander, Spain, has been established during a semester programme at ICERM at Brown University. The group around C. Beltrán is especially interested in using point processes to obtain upper bounds for minimal energy and related quantities like worst case errors in integration. This is also very suitable for a student exchange.

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