Faculty member Jörg Thuswaldner (Project 08)

Short description of two showcases of PhD Research Projects

Showcase 1. Discrepancy estimates for $\boldsymbol{\beta}$-adic Halton Sequences.
The well-known van der Corput and Halton sequences permit a generalization to number systems defined w.r.t. linear recurrences and beta-numeration. Indeed, in Ninomiya [1998] the notion of β-adic Halton sequence has been defined and it has been shown that such a sequence is a low discrepancy sequence for each Pisot number β. Later Hofer et al. [2015] proved that the $\boldsymbol{\beta}$-adic Halton sequences are equidistributed modulo $[0,1]^s$ for certain vectors $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$. Recently, Thuswaldner [2017] gave first discrepancy estimates of these sequences for the case that the entries of $\boldsymbol{\beta} = (\beta_1, \dots, \beta_s)$ are m-bonacci numbers, i.e., dominant roots of polynomials of the form $x^m - x^{m-1} - \cdots - x - 1$ $(m \geq 2)$.

There are several directions for further research on this topic that are well-suited for a PhD thesis. The first task would be to generalize the discrepancy estimate of Thuswaldner [2017] to wider classes of linear recurrences. As several obstacles have to be mastered, this should be done in two steps of increasing difficulty. In a first step the class of vectors $\boldsymbol{\beta}$ studied in Hofer et al. [2015] should be considered. In this class the language of the digit strings of the underlying linear recurrent number systems is still symmetric which makes it possible to define the van der Corput and Halton sequences by reflecting the digit expansion of a given integer at the "decimal" point. However, the dominant root β of the characteristic polynomial of the recurrences studied in Hofer et al. [2015] is a Pisot number but in general no longer a unit. To some extent, the approach of Thuswaldner [2017] can be followed. In particular, one can associate substitutions to these linear recurrences and use the underlying Rauzy fractals. However, in this more general case the Rauzy fractals are no longer subsets of the Euclidean space but live in an open subring of the adèle ring $\mathbb{A}_{\mathbb{Q}(\beta)}$. The theory of these fractals is well developed (see e.g. Minervino and Thuswaldner [2014]), and with some more technical effort they should relate the Halton sequences in question with certain rotations on these subrings. To estimate the discrepancy of these rotations generalizations of the Erdős-Turán-Koksma inequality (cf. Drmota and Tichy [1997, Theorem 1.21] for the Euclidean version) and Schlickewei's $\mathfrak{p}$-adic subspace theorem (cf. Schlickewei [1977]) could be of use. This problem should be tractable on the one side and should give the PhD candidate the opportunity to familiarize herself with a variety of deep results and theories on the other side. In a second step the PhD candidate should obtain results on Halton sequences related to β-expansions with asymmetric languages. In this case one needs to deal with the reverse language of the underlying substitutions in some way to define the appropriate Rauzy fractals in order to derive the rotation related to the Halton sequence in question.

As the discrepancy estimates of Thuswaldner [2017] are not optimal it would be interesting to gain a better understanding of the distribution of $\boldsymbol{\beta}$-adic Halton sequences that would lead to improved discrepancy estimates and to the characterization of bounded remainder sets .

Showcase 2. The sum of digit functions for nonmonic canonical number systems.
In Akiyama et al. [2008] a number system with a rational number p/q as base has been considered. Such number systems are quite different from the well-known q-ary expansions. Indeed, in Akiyama et al. [2008] the language of representations of the integers in such number systems was investigated. The authors obtained mostly negative results about this language. Notably, it is not regular which makes it hard to study. Nevertheless, they could use this

notion of number system in order to prove nontrivial results of a variant of Mahler's 3/2 problem. Moreover, Morgenbesser et al. [2013] were able to prove results on the summatory function of the sum of digits function $s_{p/q}$ of such number systems and showed that each pattern of digits occurs with the expected frequency. They also exhibited normal numbers for these number systems in the spirit of the Champernowne construction.

In a PhD thesis further normal numbers for these number systems should be constructed. Also a p/q analog of Borel's result on the genericity of normal numbers w.r.t. the Lebesgue measure should be proved based on the frequency results of Morgenbesser et al. [2013]. As observed in Scheicher et al. [2014], the concept of rational based number systems introduced in Akiyama et al. [2008] can be extended. Indeed, so-called canonical number systems (see Pethő [1991]) can be studied also for nonmonic polynomials. The case of the polynomial $qx-p$ then corresponds to the original p/q number system. Analogous problems as investigated in Akiyama et al. [2008] and Morgenbesser et al. [2013] could be studied by a PhD student in this more general context. We expect that this leads to new – "fractal" – problems and the theory on rational self-affine tiles developed in Steiner and Thuswaldner [2015] has to be used in full generality. Indeed, to get the most general results it should even be extended to reducible matrices which forms another challenge since the methods from algebraic number theory used in the irreducible case studied in Steiner and Thuswaldner [2015] are no longer applicable.

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