Faculty member Robert Tichy (Project 09)

Short description of two showcases of PhD Research Projects

Showcase 1. Diophantine Equations and Linear Recurrences
Pillai's conjecture states that for each fixed $c \geq 1$, the Diophantine equation $a^x - b^y = c, \min\{x, y\} \geq 2$, has only a finite number of positive integer solutions (a, b, x, y). This conjecture is till open; however the case $c = 1$ (Catalan's conjecture), has been proved by Mihăilescu. In the last decade various extensions of Diophantine problems involving linear recurrences were considered.

A specific topic is in the context of Pellian equation $x^2 - dy^2 = 1$ (d a positive number). This equation is completely solved (with only finitely many solutions), when x comes from certain linear recurring sequences, e.g. when $x = F_n$ is a Fibonacci number. It is an interesting problem for a PhD thesis to extend these results to more general classes of linear recurrences. It seems very natural to investigate the situation when the solutions come from k-th order linear recurrences with a dominating characteristic root. The most important case covers recurrences such that the corresponding characteristic polynomial is the minimal polynomial of a Pisot number.

Showcase 2. Complexity of absolutely normal numbers
Let $q \geq 2$ be a given positive integer. A real number $x = \sum_j \varepsilon_j q^{-j}$ is said to be normal in base q if all q-ary digital blocks of length k occur asymptotically with frequency q^{-k} :

$$\frac{1}{N} \sharp\{n \leq N : (\epsilon_n, \ldots, \epsilon_{n+k-1}) = (d_1, \ldots, d_k)\} \to \frac{1}{q^k}$$

(for all $k = 1, 2, \ldots$ and all $d_j = 0, \ldots, q-1, j = 1, \ldots k$). It is a famous open problem to show normality of numbers such as $\pi, e, \sqrt{2}$ etc. However, specific constructions for normal numbers in given base q are known, eg. Champernown's construction $x = 0,123456789101112\ldots$ in base 10. Much more difficult is the problem to find constructions of absolutely normal numbers, i.e. of numbers which are normal to all bases q. Turing (in the 1930's) raised the question whether there is a polynomial time algorithm to produce a number x which is normal in all bases $q \geq 2$. This problem was solved by Becher-Heiber-Slaman (2013), and in the following years various other algorithms were established and analyzed. It remains open to investigate the speed of convergence in the above definition of normality. It turns out that there is a trade off between the computational complexity of the constructions and the speed of convergence which can be measured by the discrepancy D_N of related sequences $(xq^n)_{n=1}^{\infty}$. A final step would be to find constructions of absolutely normal numbers such that $D_N = O(\log N/N)$, i.e. such that the corresponding sequences are low-discrepancy. It will also be a suitable project for a PhD thesis to extend these constructions to more general digital expansions such as Pisot number systems or continued fraction expansions.