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Short description of two showcases of PhD Research Projects

Subdivision means the iterative refinement of discrete data with the aim of creating a continuous limit. Linear subdivision operators are well understood, both for data on regular grids and data with polyhedral connectivity. We are interested in nonlinear, geometric, data. E.g. flight recorder data are a sequence with values in the Lie group SO_3, and diffusion tensor MRI produces images with values in the Riemannian symmetric space of 3×3 positive definite symmetric matrices. Such nonlinear, geometric data are important in many fields and are an active topic of research from PDEs to statistics [Grohs et al. 2018; Pennec 2018].

For any analysis, statistics or computation it is essential to employ operations which are *natural*, that is, invariant w.r.t. the relevant transformation groups. An important eary contribution to the systematic treatment of elementary operations in nonlinear geometries was made by the 'Symmlab' group at Stanford University, which aimed at providing software tools for symmetric spaces. Concrete suggestions for subdivision were proposed in a 2001 lecture by D. Donoho. The specific interest in subdivision comes from its usefulness for smoothing, interpolation, and multiresolution representations.

In its simple and linear form, a subdivision rule for discrete data is applied to $(p_\beta)_{\beta\in\mathbb{Z}^s}$ in $\mathbb{R}^n$ and creates denser data $(Sp_\alpha)_{\alpha\in\mathbb{Z}^s}$ by finite averages, $(Sp)_\alpha = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-N\beta}p_\beta$. We require that $\sum_\beta a_{\alpha-N\beta} = 1$. This procedure may be visualized by interpreting p as a regular sample of a curve or s-dimensional surface, and Sp is a denser sample of the same object. We are interested in situations where the sequence of iterates $p, Sp, S^2p, \ldots$ converges to a continuous limit curve resp. limit surface. The interpretation of S as averaging is key to the generalization of subdivision to the geometric setting, which is mainly based on the equivalences

$$(Sp)_\alpha = \sum\nolimits_\beta a_{\alpha-N\beta}p_\beta \iff (Sp)_\alpha = \arg\min_x \sum\nolimits_\beta a_{\alpha-N\beta}\,\mathrm{dist}(p_\beta, x)^2 \iff$$

$$(Sp)_\alpha \text{ solves } \sum\nolimits_\beta a_{\alpha-N\beta}(p_\beta - x) = 0 \iff (S_p)_\alpha = \mathbb{E}X, \text{ where } \mathbb{P}(X = p_\beta) = a_{\alpha-N\beta}.$$

Subdivision is thus defined in such metric spaces where the expression involving distances has a minimizer; and it is defined in Lie groups where the third expression is interpreted with $a \oplus v = a\exp(v)$ and $a \ominus b = \exp^{-1}(a^{-1}b)$ as substitutes for addition and subtraction. This viewpoint has been successfully exploited to obtain satisfactory results concerning smoothness of limits. The last equivalence, valid only in case of nonnegative coefficients, in the 1st phase of this DK programme has been employed by Ebner [2013, 2014] to show the following: A subdivision rule defined by coefficients $a_\gamma \geq 0$ $(\gamma \in \mathbb{Z}^s)$ in an Hadamard CAT(0) metric space produces continuous limits for any given data p, if and only if this is true in the linear case of the metric space $\mathbb{R}$.

Showcase 1. Convergence properties of geometric subdivision processes.
Existence of continuous limits is a fundamental property of subdivision processes and in the geometric (nonlinear) case depends not only on the coefficients a_γ but also on the data. It is known that a manifold subdivision scheme will produce C^k-smooth limits for all data where convergence happens, if the linear scheme with the same coefficients has this property [Grohs 2010; Duchamp et al. 2016]. Unfortunately, convergence in the general case can only be shown for *dense enough* input data. Results valid for all input data apply to special schemes like interpolatory schemes [Wallner 2014] or to special geometries like Hadamard metric spaces

[Ebner 2013, 2014]. A recent result shows convergence for all input data in the $s = 1$ case, for manifolds with nonpositive curvature, but without a sign assumption on the coefficients a_γ [Hüning and Wallner 201?]. There are still many open questions regarding convergence.

One is the obvious gap between theory and practice in situations where convergence is guaranteed only for dense enough input data. These results have been obtained by the method of proximity inequalities proposed by Wallner and Dyn [2005], which quantify the deviation of a nonlinear subdivision rule from a linear one. For multivariate data ($s \geq 2$) they probably cannot be substantially improved – this has to do with the possibility of topological features not mappable from the unit cube $[0, 1]^s$. We therefore propose to focus on the case $s = 1$.

Another open question we want to investigate concerns situations where convergence happens for all input data, if these data are multivariate ($s \geq 2$) or have more general polyhedral connectivity. So far only local results for general geometries have been obtained [Weinmann 2010]. Global results exist for the univariate case. The methods of Riemannian geometry, often founded on the seminal paper [Karcher 1977], can also be applied in the multivariate setting. They work best in case of nonpositive curvature, where the exponential mapping does not decrease distances, and to a limited extent also in the case of bounded positive curvatures. Topological features of the base space can be dealt with in a manner similar to the univariate case [Hüning and Wallner 201?]. It is also worth investigating special geometries which play an important role, such as the Riemannian symmetric space of positive definite matrices.

Besides mere convergence, further properties of the limit are interesting. Subdivision process easily yield Hölder continuity estimates, but we also want to study geometric shape properties. Oscillation can be measured against suitable submanifolds, and we propose to start this research with subdivision rules defined in terms of repeated geodesic averaging.

Showcase 2. Multiresolution representations of geometric data.
Multiresolution representation of data hinges on downscaling and upscaling procedures D, U and comparing dat p with UDp. When data $(p_\gamma)_{\gamma \in \mathbb{Z}^s}$ are interpreted as samples of an s-variate function, then Dp resp. Up are coarser resp. finer samples of the same function. Typically U is a subdivision operator, and D is defined via similar elementary operations.

Previous work on multiresolution representations of manifold-valued data concerns mostly certain conceptually easy versions of D, U. The corresponding multiresolution representation has properties similar to its linear counterpart (e.g. with regard to data regularity and coefficient decay). On the negative side, perfect reconstruction without redundancy is possible only in special cases [Grohs and Wallner 2009, 2012].

We propose to investigate multiscale representations of pixel-based and voxel-based data, and to use previous work on combinatorial singularites by Weinmann [2012] to extend results to bivariate data with polyhdral connectivity. A focus on special metric spaces is consistent with an relevant application, namely diffusion-tensor images. It will be important to study previous work (e.g. on feature detection) and compare it with methods employing only natural (invariant) operations.

The reconstruction procedure contained in any multiresolution representation of discrete data poses convergence questions. The most general question – convergence of reconstruction for arbitrary base data and detail data – contains the convergence questions posed in showcase 1 as a special case. However, a different question is both easier and more relevant: Data obtained by sampling are already known to converge upon reconstruction. We ask for bounds on perturbations (incurred by quantization or thresholding) which do not destroy that convergence.

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