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Short description of two showcases of PhD Research Projects

Showcase 1. Rotation systems of complete graphs.
For good drawings of complete graphs (simple complete topological graphs), combinatorial properties (including for example which edge pairs cross) depend only on the rotation system and can be determined without generating a concrete drawing. Surprisingly, it can be decided in polynomial time whether a rotation system of the complete graph can be realized as good drawing by checking realizability of all 6-tuples of vertices, see Kynčl [2015]. In combination with our results on the complete enumeration of all realizable rotation systems for up to nine vertices (see Ábrego et al. [2015]) this implies that it is in fact sufficient to only check realizability of all 5-tuples of vertices. An interesting question in this context is the meaning of the following complexity result: a non-realizable 4-tuple in a rotation system forces a crossing between incident edges in any drawing. Every non-realizable 5-tuple where all 4-tuples are realizable can be realized as a *semi-good drawing* (that is, a drawing where multiple crossings between edges are allowed, but incident edges do not cross). However, a recent result shows that not every rotation system where all 4-tuples are realizable admits a semi-good drawing. It is open how to decide semi-good realizability. Although it is known to be in P, the actual complexity of deciding realizability of rotation systems as good drawings is also still open. We can show that realizability of all 4-tuples can be checked in $O(n^3)$ time. Is it possible to check realizability of all 5-tuples in $o(n^5)$ time? More general, what is the complexity of deciding whether a given rotation system of the complete graph can be realized as drawing of a certain type? Similar to other results in this area, this decision problem is hard for geometric drawings, and has been shown to be in P for monotone and pseudolinear drawings in Aichholzer et al. [2015]. These questions are also connected to the topic of determining which properties can in fact be derived from rotation systems, and which properties really depend on a concrete drawing of the graph. Altogether this gives a large varyity of research direction for a prospective PhD thesis.

Showcase 2. Crossing number for complete bipartite graphs.
The search for the crossing number of the complete bipartite graph $K_{m,n}$ goes back to the 1940's, where Paul Turán was working in a forced labor camp and the problem has hence become known as Turán's *Brick Factory Problem.* In the 1950's, Zarankiewicz [1953, 1954] and K. Urbaník [1955] independently proposed the same solution for the problem, giving an upper bound construction and a lower bound proof. But some years later the proof was found to be incomplete by Kainen and Ringel (see Guy [1968]), and the former theorem became famous as Zarankiewicz's conjecture. Although many researchers have been working on this problem since then, still surprisingly little is known. For some very small values of m and n, the conjecture has been confirmed, and some improvements on lower bounds have been achieved for general drawings. Similar to the case of the complete graph, a natural restriction of the question is to consider only geometric graphs, exploiting the geometry of point sets. Attempts in this direction have been made in the thesis of Vogtenhuber [2011] – among others generalising k-edges to the bipartite setting and considering the even more restricted setting of geometric drawings where the two subsets of the point set are linearly separable – but did not yet yield significant results. Note that, in contrast to Hill's drawings of the complete graph, Zarankiewicz's drawings of the complete bipartite graph are also straight-line drawings. This

implies that if Zarankiewicz's conjecture is true, there is no separation between the crossing numbers of straight-line and good drawings in the bipartite case.

Recently, in collaboration with Gelasio Salazar, new observations on the separated case have been obtained. Using circular allowable sequences of point sets a purely combinatorial setting of the separated case can be obtained. Considering (weighted) flips of the projected points, a direct relation to the crossing number has been shown. This enables investigations of even more restricted cases of Zarankiewicz's construction and allows to show lower bounds in these cases. After about 60 years of the conjecture, this approach opens an interesting path with high probability of success. We expect this to provide extra motivation for a PhD student to be involved in progress on a long standing open problem.

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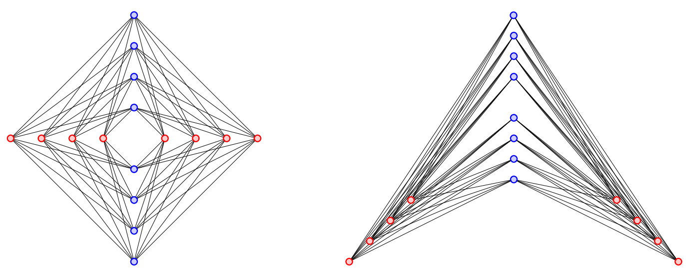

FIGURE 1. Conjectured crossing-minimizing drawings: Zarankiewicz's construction for $K_{m,n}$ (left) and the linearly separated version (right).