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Short description of two showcases of PhD Research Projects

Showcase 1. Critical random (planar) graphs.
Since the foundation of the theory of random graphs by Erdős and Rényi, various interesting models of random graphs have been introduced and investigated, and exciting similarities and differences to the Erdős–Rényi random graph have been discovered.

One of the striking results on the Erdős-Rényi random graph $G(n,m)$, a graph taken uniformly at random from the set of all vertex-labelled graphs on vertex set $[n]$ with exactly m edges, is the dramtic change in its structure around the critical period when $m = \frac{n}{2} + O(n^{2/3})$, where the graph typically changes from being *planar* with 'small' components of logarithmic order to being *non-planar* with a 'giant' component of linear order. This naturally motivates the question of exactly how the *genus* of $G(n,m)$ behaves at/after the critical period. Here the genus of a graph is the smallest value of g for which the graph can be embedded on a surface of genus g without any crossing edges.

As a natural model of a random graph *with constraints*, random planar graphs have been the subject of much work [Giménez and Noy 2009; Kang and Loebl 2009; Kang and Łuczak 2012; McDiarmid et al. 2005]. We let $P(n,m)$ denote a graph taken uniformly at random from the set of all vertex-labelled graphs on vertex set $[n]$ with exactly m edges that *can be embedded on the sphere without crossing edges*. Such a graph is called a random planar graph. Kang and Łuczak [2012] showed that in the evolution of a random planar graph $P(n,m)$, there are (perhaps rather surprisingly) two critical periods. The first critical period in $P(n,m)$ occurs when the giant component is formed, which happens at $m = \frac{n}{2} + O(n^{2/3})$, and this coincides with that of $G(n,m)$. The second critical period in $P(n,m)$ is when the giant component covers nearly all vertices, which happens at $m = n + O(n^{3/5})$.

Aldous [1997] proved that when normalised by $n^{-2/3}$, the asymptotic joint distribution of the component orders in the *critical random graph* $G(n,m)$ (i.e. with $m = \frac{n}{2} + O(n^{2/3})$) is the same as the joint distribution of the excursion lengths of a particular Brownian motion. However, because $G(n,m)$ starts to become non-planar when $m = \frac{n}{2} + O(n^{2/3})$, it is not the case that the orders of the components of $P(n,m)$ must have the same distribution as for $G(n,m)$. Hence, it would be extremely interesting to investigate the joint distribution of the orders of the largest components of the *critical random planar graph* $P(n,m)$ (in the first critical period) and to discover how this differs from the critical random graph $G(n,m)$.

It is known that the *local structure* of $G(n,m)$ with $m = \alpha\frac{n}{2}$ for $\alpha \in (1,\infty)$ converges to that of a *Galton-Watson tree* with offspring distribution $\mathrm{Po}(\alpha)$ in the sense of Benjamini-Schramm local weak convergence [Benjamini and Schramm 2001]. In particular, *whp* (with high probability, meaning with probability tending to 1 as $n \to \infty$), the order of the largest component in $G(n, \alpha\frac{n}{2})$ is $(\beta + o(1))n$, where β is the unique positive solution of the equation $1 - \beta = \exp(-\alpha\beta)$. However, due to the topological constraint imposed on $P(n,m)$ (i.e. genus $g = 0$), the exploration of components in $P(n,m)$ via a simple Galton-Watson tree is not possible. Moreover, Kang and Łuczak [2012] showed that when $m = \alpha\frac{n}{2}$ for $\alpha \in (1,2)$, whp the order of the largest component in $P(n,m)$ is $(\alpha - 1 + o(1))n$. Hence, this naturally raises the question of whether the local structure of a critical random planar graph $P(n,m)$ (when $\alpha \to 1$ or $\alpha \to 2$) can be described as a 'simple natural' tree in terms of the Benjamini-Schramm local weak convergence.

Showcase 2. Random graphs on surfaces with positive genus.
As a generalisation of random planar graphs, random graphs on surfaces with positive genus have been extensively studied e.g. in [Chapuy et al. 2011; Kang et al. 2018; McDiarmid 2008].

Note that the information on the genus of the Erdős-Rényi random graph $G(n, m)$ provides a way to investigate random graphs on surfaces. We let $S_g(n, m)$ denote a graph taken uniformly at random from the set of all vertex-labelled graphs on vertex set $[n]$ with exactly m edges and *with genus at most* g. If $G(n, m)$ can be shown to have genus at most g whp for some specified values of $g = g(n)$ and $m = m(n)$, then this would immediately imply that for these values the random graphs $G(n, m)$ and $S_g(n, m)$ must essentially behave in the same way.

The recent study of random graphs on surfaces [Chapuy et al. 2011; Kang et al. 2018; McDiarmid 2008] indicates that when the genus g is a *constant* independent of n, the actual value of g has little impact. For example, Kang et al. [2018] proved that $S_g(n, m)$ for constant g exhibits two critical periods, similarly to $P(n, m)$. Furthermore, it follows from [Kang et al. 2018] that at the critical periods the graph obtained from $S_g(n, m)$ by deleting the giant component resembles a critical random planar graph $P(n, m)$. Hence, any insight into the behaviour of a critical random planar graph $P(n, m)$ could lead to further knowledge about a critical random graphs on surfaces $S_g(n, m)$.

However, when the genus grows as a function of n, this can certainly have substantial influence on certain structural properties of $S_g(n, m)$. When the genus $g = g(n)$ grows 'sufficiently slowly', it would be natural to expect that the structure and order of the components of $S_g(n, m)$ evolve in a similar manner as in the constant genus case. On the other hand, when g grows 'fast enough', the exhibited behaviour should instead mimic $G(n, m)$. The central questions here are how slowly or how fast the genus $g = g(n)$ should grow, so as to cause and result in such different structural behaviours of $S_g(n, m)$.

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