Lacunary series and stable distributions

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Dedicated to Professor Paul Deheuvels on the occasion of his 65th birthday

Abstract

By well known results of probability theory, any sequence of random variables with bounded second moments has a subsequence satisfying the central limit theorem and the law of the iterated logarithm in a randomized form. In this paper we give criteria for a sequence (X_n) of random variables to have a subsequence (X_{n_k}) whose weighted partial sums, suitably normalized, converge weakly to a stable distribution with parameter $0 < \alpha < 2$.

1 Introduction

It is known that sufficiently thin subsequences of general r.v. sequences behave like i.i.d. sequences. For example, Chatterji [7], [8] and Gaposhkin [11], [12] proved that if a sequence (X_n) of r.v.'s satisfies $\sup_n EX_n^2 < \infty$, then one can find a subsequence (X_{n_k}) and r.v.'s X and $Y \geq 0$ such that

$$\frac{1}{\sqrt{N}} \sum_{k \leq N} (X_{n_k} - X) \xrightarrow{d} N(0, Y) \tag{1.1}$$

and

$$\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \sum_{k \leq N} (X_{n_k} - X) = Y^{1/2} \qquad \text{a.s.,} \tag{1.2}$$

where $N(0, Y)$ denotes the distribution of the r.v. $Y^{1/2}\zeta$ where ζ is an $N(0,1)$ r.v. independent of Y. Komlós [15] proved that under $\sup_n E|X_n| < \infty$ there exists a subsequence (X_{n_k}) and an integrable r.v. X such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} X_{n_k} = X \qquad \text{a.s.}$$

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and Chatterji [6] showed that under $\sup_n E|X_n|^p < \infty$, $0 < p < 2$ the conclusion of the previous theorem can be changed to

$$\lim_{N\to\infty} \frac{1}{N^{1/p}} \sum_{k=1}^{N} (X_{n_k} - X) = 0 \qquad \text{a.s.}$$

for some X with $E|X|^p < \infty$. Note the randomization in all these examples: the role of the mean and variance of the subsequence (X_{n_k}) is played by random variables X, Y. On the basis of these and several other examples, Chatterji [9] formulated the following heuristic principle:

Subsequence Principle. *Let T be a probability limit theorem valid for all sequences of i.i.d. random variables belonging to an integrability class L defined by the finiteness of a norm $\| \cdot \|_L$. Then if (X_n) is an arbitrary (dependent) sequence of random variables satisfying $\sup_n \|X_n\|_L < +\infty$ then there exists a subsequence (X_{n_k}) satisfying T in a mixed form.*

In a profound paper, Aldous [1] proved the validity of this principle for all limit theorems concerning the almost sure or distributional behavior of a sequence of functionals $f_k(X_1, X_2, \ldots)$ of a sequence (X_n) of r.v.'s. Most "usual" limit theorems belong to this class; for precise formulations, discussion and examples we refer to [1]. On the other hand, the theory does not cover functionals f_k containing parameters (as in weighted limit theorems) or allows limit theorems to involve other type of uniformities. Such uniformities play an important role in analysis. For example, if from a sequence (X_n) of r.v.'s with finite p-th moments ($p \geq 1$) one can select a subsequence (X_{n_k}) such that

$$K^{-1} \left(\sum_{i=1}^{N} a_i^2 \right)^{1/2} \leq \Big\| \sum_{i=1}^{N} a_i X_{n_i} \Big\|_p \leq K \left(\sum_{i=1}^{N} a_i^2 \right)^{1/2}$$

for some constant $0 < K < \infty$, for every $N \geq 1$ and every $(a_1, \ldots, a_N) \in \mathbb{R}^N$, then the subspace of L^p spanned by (X_n) contains an subspace isomorphic to Hilbert space. Such embedding arguments go back to the classical paper of Kadec and Pelczynski [14] and play an important role in Banach space theory, see e.g. Dacunha-Castelle and Krivine [10], Aldous [2]. In the theory of orthogonal series and in Banach space theory we frequently need subsequences (f_{n_k}) of a sequence (f_n) such that $\sum_{k=1}^{\infty} c_k f_{n_k}$ converges a.e. or in norm, after any permutation of its terms, for a class of coefficient sequences (a_k). Here we need uniformity both over a class of coefficient sequences (a_k) and over all permutations of the terms of the series. A number of uniform limit theorems for subsequences have been proved by ad hoc arguments. Révész [18] showed that for any sequence (X_n) of r.v.'s satisfying $\sup_n EX_n^2 < \infty$ one can find a subsequence (X_{n_k}) and a r.v. X such that $\sum_{k=1}^{\infty} a_k (X_{n_k} - X)$ converges a.s. provided $\sum_{k=1}^{\infty} a_k^2 < \infty$. Under $\sup_n \|X_n\|_\infty < +\infty$, Gaposhkin [11] showed that there exists a subsequence (X_{n_k}) and r.v.'s X and $Y \geq 0$ such that for any real sequence (a_k)

satisfying the uniform asymptotic negligibility condition

$$\max_{1\le k\le N} |a_k| = o(A_N), \qquad A_N = \left(\sum_{k=1}^N a_k^2\right)^{1/2} \tag{1.3}$$

we have

$$\frac{1}{A_N}\sum_{k\le N} a_k(X_{n_k}-X) \xrightarrow{d} N(0,Y) \tag{1.4}$$

and for any real sequence (a_k) satisfying the Kolmogorov condition

$$\max_{1\le k\le N} |a_k| = o(A_N/(\log\log A_N)^{1/2}) \tag{1.5}$$

we have

$$\frac{1}{(2A_N\log\log A_N)^{1/2}}\sum_{k\le N} a_k(X_{n_k}-X) = Y^{1/2} \qquad \text{a.s.} \tag{1.6}$$

For a fixed coefficient sequence (a_k) the above results follow from Aldous' general theorems, but the subsequence (X_{n_k}) provided by the proofs depends on (a_k) and to find a subsequence working for all (a_k) simultaneously requires a uniformity which is, in general, not easy to establish and it can fail in important situations. (See Guerre and Raynaud [13] for a natural problem where uniformity is not valid.) In [1], Aldous used an equicontinuity argument to prove a permutation-invariant version of the theorem of Révész above, implying that every orthonormal system (f_n) contains a subsequence (f_{n_k}) which, using the standard terminology, is an *unconditional convergence system.* This has been a long standing open problem in the theory of orthogonal series (see Uljanov [19], p. 48) and was first proved by Komlós [16]. In [3] we used the method of Aldous to prove extensions of the Kadec-Pelczynski theorem, as well as selection theorems for almost symmetric sequences. The purpose of the present paper is to use a similar technique to prove an uniform limit theorem of probabilistic importance, namely the analogue of Gaposhkin's uniform CLT (1.3)–(1.4) in the case when the limit distribution of the normed sum is a stable law with parameter $0<\alpha<2$. To formulate our result, we need some definitions. Using the terminology of [5], call the sequence (X_n) of r.v.'s *determining* if it has a limit distribution relative to any set A in the probability space with $P(A)>0$, i.e. for any $A\subset\Omega$ with $P(A)>0$ there exists a distribution function F_A such that

$$\lim_{n\to\infty} P(X_n<t\mid A) = F_A(t)$$

for all continuity points t of F_A. By an extension of the Helly-Bray theorem (see [5]), every tight sequence of r.v.'s contains a determining subsequence. Hence in studying the asymptotic behavior of thin subsequences of general tight sequences we can assume without loss of generality that our original sequence (X_n) is determining. As is shown in [1], [5], for any determining sequence (X_n) there exists a random measure μ (i.e. a measurable map from the underlying probability space $(\Omega,\mathcal{F},\mathcal{P})$ to the space $\mathcal{M}$ of probability measures on $\mathbb{R}$ equipped with the Prohorov metric) such that for any A with $P(A) > 0$ and any continuity point t of F_A we have

$$F_A(t) = E_A(\mu(-\infty, t)), \tag{1.7}$$

where E_A denotes conditional expectation given A. We call μ the *limit random measure* of (X_n). The situation concerning the unweighted CLT for lacunary sequences can be summarized by the following theorem.

Theorem 1.1 *Let (X_n) be a determining sequence of r.v.'s with limit random measure μ. Then there exists a subsequence (X_{n_k}) satisfying, together with all of its subsequences, the CLT (1.1) with suitable r.v.'s X and $Y \geq 0$ if and only if*

$$\int_{-\infty}^{\infty} x^2 d\mu(x) < \infty \qquad a.s. \tag{1.8}$$

The sufficiency part of the theorem is contained in Aldous'general subsequence theorems in [1]; the necessity was proved in our recent paper [4]. Note that the condition for the CLT for lacunary subsequences of (X_n) is given in terms of the limit random measure of (X_n) and this condition is the exact analogue of the condition in the i.i.d. case, only the common distribution of the i.i.d. variables is replaced by the limit random measure. Note also that the existence of second moments of (X_n) (or the existence of any moments) is not necessary for the conclusion of Theorem 1.1.

In this paper we investigate the analogous question in case of a nonnormal stable limit distribution, i.e. the question under what conditions a sequence (X_n) of r.v.'s has a subsequence (X_{n_k}) whose weighted partial sums, suitably normalized, converge weakly to an α-stable distribution, $0 < \alpha < 2$. Let, for $c > 0$ and $0 < \alpha < 2$, $G_{\alpha,c}$ denote the distribution function with characteristic function $\exp(-c|t|^\alpha)$ and let $S = S(\alpha, c)$ denote the class of distributions on $\mathbb{R}$ with characteristic function φ satisfying

$$\varphi(t) = 1 - c|t|^\alpha + o(|t|^\alpha) \qquad \text{as } t \to 0. \tag{1.9}$$

Our main result is

Theorem 1.2 *Let $0 < \alpha < 2$, $c > 0$ and let (X_n) be a determining sequence of r.v.'s with limit random measure μ. Assume that $\mu \in S(\alpha, c)$ with probability 1. Then there exists a subsequence (X_{n_k}) such that for any real sequence (a_k) satisfying*

$$\max_{1\leq k\leq N} |a_k| = o(A_N), \quad A_N = \left(\sum_{k=1}^N |a_k|^\alpha\right)^{1/\alpha} \tag{1.10}$$

we have

$$A_N^{-1} \sum_{k=1}^N a_k X_{n_k} \xrightarrow{d} G_{\alpha,c}.$$

A simple calculation with characteristic functions, similar to the proof of Lemma 2.1 below, shows that if (Y_n) is sequence of i.i.d. random variables with a characteristic function φ satisfying (1.9) and (a_n) is a real sequence satisfying (1.10), then

$$A_N^{-1} \sum_{k=1}^{N} a_k Y_k \xrightarrow{d} G_{\alpha,c}. \tag{1.11}$$

Thus, just like in the case of Theorem 1.1, the basic assumption in Theorem 1.2 is a condition close to the classical i.i.d. condition, only it is required for almost all realizations of the limit random measure μ. It is natural to conjecture that the assumption $\mu \in S(\alpha, c)$ a.s. in Theorem 1.2 can be weakened to the assumption that with probability 1, μ is in the domain of normal attraction of the stable law with characteristic function $\exp(-c|t|^\alpha)$ and this condition is also necessary, but this remains open.

2 Proof of Theorem 1.2

Given $\mu_1, \mu_2 \in S(\alpha, c)$ with characteristic functions φ_1, φ_2, we can write

$$\varphi_1(t) = 1 - c|t|^\alpha + \beta_1(t)|t|^\alpha, \qquad \varphi_2(t) = 1 - c|t|^\alpha + \beta_2(t)|t|^\alpha, \quad t \in \mathbb{R} \tag{2.1}$$

where $\lim_{t\to 0} \beta_1(t) = \lim_{t\to 0} \beta_2(t) = 0$. Define a metric ρ on $S(\alpha, c)$ by

$$\rho(\mu_1, \mu_2) = \sup_{|t|\le 1} |\beta_1(t) - \beta_2(t)|. \tag{2.2}$$

Lemma 2.1 *Let $\mu_1, \mu_2 \in S(\alpha, c)$ satisfying (1.9), let $Z_1, \dots, Z_n$ and $Z_1^*, \dots, Z_n^*$ be i.i.d. sequences with respective distributions μ_1, μ_2. Let $(a_1, \dots, a_n) \in \mathbb{R}^n$, $A_n = \left(\sum_{k=1}^n |a_k|^\alpha\right)^{1/\alpha}$. Then for $|t| \le 1$*

$$\left| E \exp\left(it A_n^{-1} \sum_{k=1}^n a_k Z_k \right) - E \exp\left(it A_n^{-1} \sum_{k=1}^n a_k Z_k^* \right) \right| \le C\rho(\mu_1, \mu_2) \tag{2.3}$$

where ρ is defined by (2.2) and C is a constant.

Proof. Letting φ_1, φ_2 denote the characteristic function of the Z_k's resp. Z_k^*'s and using (2.1), (1.10) and $\log(1+x) = x + O(x^2)$ for $|x| \le 1$, the first expectation in (2.3) equals for $|t| \le 1$

$$\prod_{k=1}^n \varphi_1(ta_k/A_n) = \exp\left(\sum_{k=1}^n \log \varphi_1(ta_k/A_n) \right)$$
$$= \exp\left(\sum_{k=1}^n \left[(\varphi_1(ta_k/A_n) - 1) + O(1)(\varphi_1(ta_k/A_n) - 1)^2 \right] \right)$$

$$= \exp\left(-\sum_{k=1}^{n}\left[\,|ta_k/A_n|^{\alpha}(c-\beta_1(ta_k/A_n)) + O(|ta_k/A_n|^{2\alpha})(c-\beta_1(ta_k/A_n))^2\right]\right).$$

Writing a similar formula for the second expectation in (2.3) and subtracting, we get Lemma 1 by simple calculations, using $|ta_k/A_n| \le 1$ and $\sum_{k=1}^{n} |ta_k/A_n|^{\alpha} = 1$.

Let $(\Omega, \mathcal{F}, P)$ be the probability space of the X_n's and $\mathbf{X} = (X_1, X_2, \ldots)$; let further μ be the limit random measure of (X_n). Let (Y_n) be a sequence of r.v.'s on $(\Omega, \mathcal{F}, P)$ such that, given $\mathbf{X}$ and μ, the r.v.'s $Y_1, Y_2, \ldots$ are conditionally i.i.d. with distribution μ, i.e.,

$$P(Y_1 \in B_1, \ldots, Y_k \in B_k | \mathbf{X}, \mu) = \prod_{i=1}^{k} P(Y_i \in B_i | \mathbf{X}, \mu) \quad \text{a.s.} \tag{2.4}$$

$$P(Y_j \in B | \mathbf{X}, \mu) = \mu(B) \quad \text{a.s.} \tag{2.5}$$

for any j, k and Borel sets $B, B_1, \ldots, B_k$ on the real line. Such a sequence (Y_n) always exists after a suitable enlargement of the probability space.

Lemma 2.2 *For every $\sigma(\mathbf{X})$-measurable* r.v. Z *and any $j \ge 1$ we have*

$$(X_n, Z) \stackrel{d}{\longrightarrow} (Y_j, Z).$$

Given probability measures ν_n, ν on the Borel sets of a separable metric space (S, d) we say, as usual, that $\nu_n \stackrel{d}{\longrightarrow} \nu$ if

$$\int_S f(x) d\nu_n(x) \longrightarrow \int_S f(x) d\nu(x) \quad \text{as} \quad n \to \infty \tag{2.6}$$

for every bounded, real valued continuous function f on S. (2.6) is clearly equivalent to

$$Ef(Z_n) \longrightarrow Ef(Z) \tag{2.7}$$

where Z_n, Z are r.v.'s valued in (S, d) (i.e. measurable maps from some probability space to (S, d)) with distribution ν_n, ν.

Lemma 2.3 *(see [17]). Let (S, d) be a separable metric space and let $\nu, \nu_1, \nu_2, \ldots$ be probability measures on the Borel sets of (S, d) such that $\nu_n \stackrel{d}{\longrightarrow} \nu$. Let $\mathcal{G}$ be a class of real valued functions on (S, d) such that*

(a) $\mathcal{G}$ is locally equicontinuous, i.e. for for every $\varepsilon > 0$ and $x \in S$ there is a $\delta = \delta(\varepsilon, x) > 0$ such that $y \in S$, $d(x, y) \le \delta$ imply $|f(x) - f(y)| \le \varepsilon$ for every $f \in \mathcal{G}$.

(b) There exists a continuous function $g \ge 0$ on S such that $|f(x)| \le g(x)$ for all $f \in \mathcal{G}$ and $x \in S$ and

$$\int_S g(x)d\nu_n(x) \longrightarrow \int_S g(x)d\nu(x) \ (<\infty) \quad as \quad n\to\infty. \tag{2.8}$$

Then

$$\int_S f(x)d\nu_n(x) \longrightarrow \int_S f(x)d\nu(x) \quad as \quad n\to\infty \tag{2.9}$$

uniformly in $f\in\mathcal{G}$.

Assume now that (X_n) satisfies the assumptions of Theorem 1.2, fix $t\in\mathbb{R}$ and for any $n\ge 1$, $(a_1,\dots,a_n)\in\mathbb{R}^n$ let

$$\psi(a_1,\dots,a_n) = E\exp\left(itA_n^{-1}\sum_{k=1}^n a_kY_k\right), \tag{2.10}$$

where $A_n = (\sum_{k=1}^n |a_k|^\alpha)^{1/\alpha}$ and (Y_k) is the sequence of r.v.'s defined before Lemma 2.2. We show that for any $\varepsilon>0$ there exists a sequence $n_1<n_2<\cdots$ of integers such that

$$(1-\varepsilon)\psi(a_1,\dots,a_k) \le E\exp\left(itA_k^{-1}\sum_{i=1}^k a_iX_{n_i}\right) \le (1+\varepsilon)\psi(a_1,\dots,a_k) \tag{2.11}$$

for all $k\ge 1$ and all (a_k) satisfying (1.10). To begin with, let us recall that by the crucial assumption of Theorem 1.2, the limit random measure μ belongs to $S(\alpha,c)$ with probability 1, i.e. for almost all ω the characteristic function of μ satisfies (1.9). The $o(|t|^\alpha)$ in (1.9) depends on ω, but by a standard measure theoretic argument, for any $\delta>0$ there exists a set $\Omega'\subset\Omega$ with probability $\ge 1-\delta$ such that the $o(|t|^\alpha)$ in (1.9) is uniform for $\omega\in\Omega'$. Thus using a diagonal argument we can assume, without loss of generality, that the characteristic function of the limit random measure μ satisfies (1.9) uniformly. To construct n_1 we set

$$Q(\mathbf{a},n,\ell) = \exp\left(itA_\ell^{-1}(a_1X_n + a_2Y_2 + \cdots + a_\ell Y_\ell)\right)$$
$$R(\mathbf{a},\ell) = \exp\left(itA_\ell^{-1}(a_1Y_1 + a_2Y_2 + \cdots + a_\ell Y_\ell)\right)$$

for every $n\ge 1$, $\ell\ge 2$ and $\mathbf{a}=(a_1,\dots,a_\ell)\in R^\ell$. We show that

$$E\left\{\frac{Q(\mathbf{a},n,\ell)}{\psi(\mathbf{a})}\right\} \longrightarrow E\left\{\frac{R(\mathbf{a},\ell)}{\psi(\mathbf{a})}\right\} \quad \text{as } n\to\infty \quad \text{uniformly in } \mathbf{a},\ell. \tag{2.12}$$

(The right side of (2.12) equals 1.) To this end we recall that, given $\mathbf{X}$ and μ, the r.v.'s $Y_1,Y_2,\dots$ are conditionally i.i.d. with common conditional distribution μ and thus, given $\mathbf{X},\mu$ and Y_1, the r.v.'s $Y_2,Y_3,\dots$ are conditionally i.i.d. with distribution μ. Thus

$$E\big(Q(\mathbf{a},n,\ell)|\mathbf{X},\mu\big) = g^{\mathbf{a},\ell}(X_n,\mu) \tag{2.13}$$

and

$$E\big(R(\mathbf{a},\ell)|\mathbf{X},\mu,Y_1\big) = g^{\mathbf{a},\ell}(Y_1,\mu), \tag{2.14}$$

where

$$g^{\mathbf{a},\ell}(u,\nu) = E\exp\left(itA_\ell^{-1}\left(a_1 u + \sum_{i=2}^{\ell} a_i\xi_i^{(\nu)}\right)\right) \qquad (u\in\mathbb{R}^1\ ,\ \nu\in S)$$

and $(\xi_n^{(\nu)})$ is an i.i.d. sequence with distribution ν. Integrating (2.13) and (2.14), we get

$$E\big(Q(\mathbf{a},n,\ell)\big) = Eg^{\mathbf{a},\ell}(X_n,\mu) \tag{2.15}$$

$$E\big(R(\mathbf{a},\ell)\big) = Eg^{\mathbf{a},\ell}(Y_1,\mu) \tag{2.16}$$

and thus (2.12) is equivalent to

$$E\frac{g^{\mathbf{a},\ell}(X_n,\mu)}{\psi(\mathbf{a})} \longrightarrow E\frac{g^{\mathbf{a},\ell}(Y_1,\mu)}{\psi(\mathbf{a})} \quad \text{as} \quad n\to\infty, \quad \text{uniformly in} \quad \mathbf{a},\ell. \tag{2.17}$$

We shall derive (2.17) from Lemma 2.1 and Lemma 2.3. Recall that ρ is a metric on $S = S(\alpha,c)$; clearly convergence in this metric implies ordinary weak convergence of probability measures, i.e. convergence in the Prohorov metric π. Thus the limit random measure μ, which is a random variable taking values in (S,π), can be also regarded as a random variable taking values in (S,ρ). Also, μ is clearly $\sigma(\mathbf{X})$ measurable and thus $(X_n,\mu) \xrightarrow{d} (Y_1,\mu)$ by Lemma 2.2. Hence, (2.17) will follow from Lemma 2.3 (note the equivalence of (2.6) and (2.7)) if we show that the class of functions

$$\left\{\frac{g^{\mathbf{a},\ell}(t,\nu)}{\psi(\mathbf{a})}\right\} \tag{2.18}$$

defined on the product metric space $(\mathbb{R}\times S\ ,\ \lambda\times\rho)$ (λ denotes the ordinary distance on $\mathbb{R}$) satisfies conditions (a),(b) of Lemma 2.3. To see the validity of (a), note that since the characteristic function of μ satisfies (1.9) uniformly, the proof of Lemma 2.1 shows that there exists an integer n_0 and a positive constant c_0 such that $\psi(\mathbf{a}) \ge c_0$ for $n\ge n_0$ and all (a_k). Thus the validity of (a) follows from Lemma 2.1; the validity of (b) is immediate from $|g^{\mathbf{a},\ell}(u,\nu)| \le 1$. We thus proved relation (2.17) and thus also (2.12), whence it follows (note again that the right side of (2.12) equals 1) that

$$\psi(\mathbf{a})^{-1}E\exp\big(itA_\ell^{-1}(a_1X_n + a_2Y_2 + \cdots + a_\ell Y_\ell)\big) \longrightarrow 1 \tag{2.19}$$

as $n\to\infty$, uniformly in $\ell,\mathbf{a}$. Hence given $\varepsilon>0$, we can choose n_1 so large that

$$|E\exp\big(itA_\ell^{-1}(a_1X_{n_1} + a_2Y_2 + \cdots + a_\ell Y_\ell)\big) - E\exp(itA_\ell^{-1}(a_1Y_1 + a_2Y_2 + \cdots + a_\ell Y_\ell))|$$
$$\le \frac{\varepsilon}{2}\psi(a_1,\ldots,a_\ell)$$

for every $\ell,\mathbf{a}$. This completes the first induction step.

Assume now that $n_1,\ldots,n_{k-1}$ have already been chosen. Exactly in the same way as we proved (2.19), it follows that for $\ell>k$

$$\psi(\mathbf{a})^{-1}E\exp\big(itA_\ell^{-1}(a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kX_n + a_{k+1}Y_{k+1} + \cdots + a_\ell Y_\ell)\big)$$

$$\longrightarrow \psi(\mathbf{a})^{-1} E \exp\left(itA_\ell^{-1}(a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kY_k + \cdots + a_\ell Y_\ell)\right) \quad \text{as} \quad n \to \infty$$

uniformly in $\mathbf{a}$ and ℓ. Hence we can choose $n_k > n_{k-1}$ so large that

$$\begin{aligned}
&E \exp\left(itA_\ell^{-1}(a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kX_{n_k} + a_{k+1}Y_{k+1} + \cdots + a_\ell Y_\ell)\right) \\
&\qquad - E \exp\left(itA_\ell^{-1}(a_1X_{n_1} + \cdots + a_{k-1}X_{n_{k-1}} + a_kY_k + \cdots + a_\ell Y_\ell)\right) \\
&\le \frac{\varepsilon}{2^k}\psi(a_1, \dots, a_\ell)
\end{aligned}$$

for every $(a_1, \dots, a_\ell) \in R^\ell$ and $\ell > k$. This completes the k-th induction step; the so constructed sequence (n_k) obviously satisfies

$$\begin{aligned}
&E \exp\left(itA_\ell^{-1}(a_1X_{n_1} + \cdots + a_\ell X_{n_\ell})\right) - E \exp\left(itA_\ell^{-1}(a_1Y_1 + \cdots + a_\ell Y_\ell)\right) \\
&\le \varepsilon\psi(a_1, \dots, a_\ell)
\end{aligned}$$

for every $\ell \ge 1$ and $(a_1, \dots, a_\ell) \in R^\ell$, i.e. (2.11) is valid.

To complete the proof of our theorem, it suffices to show that for any $t \in \mathbb{R}$ and any real sequence (a_k) satisfying (1.10) we have

$$E \exp\left(itA_k^{-1}\sum_{j=1}^k a_jY_j\right) \longrightarrow \exp(-c|t|^\alpha) \qquad \text{as } k \to \infty. \tag{2.20}$$

Together with (2.11) this implies that for any $\varepsilon > 0$ and $t \in \mathbb{R}$ there exists a sequence $n_1 < n_2 < \dots$ of positive integers (depending on ε and t) such that

$$\left| E \exp\left(itA_k^{-1}\sum_{j=1}^k a_jX_{n_j}\right) - \exp(-c|t|^\alpha)\right| < \varepsilon$$

for any $k \ge 1$ and any (a_k) satisfying (1.10). By a diagonal argument this shows that there exists a sequence $m_1 < m_2 < \dots$ such that

$$E \exp\left(itA_k^{-1}\sum_{j=1}^k a_jX_{m_j}\right) \longrightarrow \exp(-c|t|^\alpha)$$

for any rational $t \in \mathbb{R}$ and any (a_k) satisfying (1.10), which implies that

$$A_k^{-1}\sum_{j=1}^k a_jX_{m_j} \xrightarrow{d} G_{\alpha,c}$$

which is the conclusion of Theorem 1.2. To verify (2.20), let us note that conditionally on $(\mathbf{X}, \mu)$, Y_j are i.i.d. with conditional characteristic function φ satisfying (1.9), which implies (see e.g. the proof of Lemma 2.1) that setting $S_k = \sum_{j=1}^k a_jY_j$,

$$E \exp\left(itA_k^{-1}S_k | \mathbf{X}, \mu\right) \longrightarrow \exp(-c|t|^\alpha). \tag{2.21}$$

Integrating the last relation and using the dominated convergence theorem we get (2.20), completing the proof of Theorem 1.2.

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