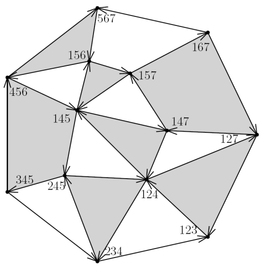
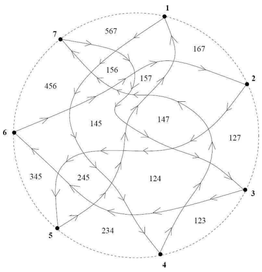
Faculty member Karin Baur (Project 12)

Short description of two showcases of PhD Research Projects

The interplay between algebra, geometry and combinatorics is a major focus of my current research. We take surfaces obtained from gluing polygons and use them to define and explore algebraic structures such as cluster categories, a young and thriving field with links to many other areas in Mathematics. The interactions between geometry and algebra are illustrated by the classical theorems of Pythagoras and Ptolemy. They characterise geometric properties of a polygon via algebraic equations. Ptolemy's theorem gives us information on one diagonal in the quadrilateral in terms of the other. We call the relation between the two diagonals a flip and its algebraic effect mutation, a key notion in cluster categories. One of our aims is to determine this mutation in associated categories, thus using elementary geometry to approach involved algebraic structures. Since mutation is a phenomenon present in many research areas in Pure Mathematics, understanding various aspects of mutation will illuminate and advance other areas.

Showcase 1. Dimer algebras on surfaces.
I propose to use dimer models with boundary to define and explore cluster structures on arbitrary surfaces. A *dimer model* is an oriented graph embedded in a surface such that its complement is a union of disks. See the figure on the left. This data naturally associates an algebra to a dimer model, with a basis given by paths in the graph and multiplication arising from concatenation of paths. The orientations of the faces encode relations for this *dimer algebra*. Sources for dimer algebras are abundant, most relevant for the proposal are dimers models arising from Postnikov's strand diagrams on disks, Postnikov [2006].

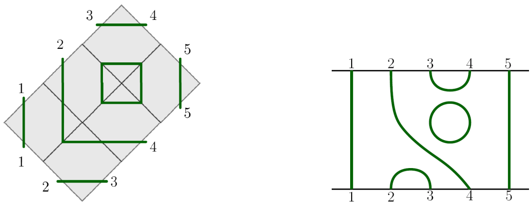




In this project, a student will develop a notion of dimer algebras on (oriented) surfaces and study their boundary algebras. The idea is to start from the work Baur et al. [2016] for the Grassmannian where the dimer algebra arises from strand diagrams on the disk (see the figure on the right). From this algebra, we obtain the *boundary algebra* by taking the idempotent subalgebra with respect to the sum of the idempotents of boundary vertices. The goal is to find a category of modules for this algebra and study its relation to known cluster categories. We expect that the projective-injective objects in this category correspond to boundary segments of the surface.

Showcase 2. Decorated tilings and mutations.
Mutation is the key notion in cluster theory, most prominent is the exchange notion arising from flips in triangulations (Fomin and Zelevinsky [2002], Fomin et al. [2008]), relying on Hatcher's result Hatcher [1991] that for any pair of triangulations of a given surface, there exists a sequence of flips connecting them.

Triangulations of surfaces are essential in defining cluster categories. Generalizing this, quadrangular tilings of surfaces are models for higher cluster categories. The interplay between surface geometry and algebraic questions is further illustrated by the use of decorated tilings in the context of bases for (coloured) Temperley-Lieb algebras, cf. Marsh and Martin [2011], di Francesco [1998], as illustrated in the figure.



The decorations here are pairs of non-crossing curves, linking midpoints of neighboured edges. The student will start from here and use a new kind of decorated tilings. The idea is to work with a pair of oriented three-legged trees on rhombic tilings, connecting mid-points of three successive edges of the tile. I expect that this will result in Kuperberg spiders, Kuperberg [1996], which are essential in the representation theory of $U_q(\mathfrak{sl}_3)$. The student will explore this with the goal to provide bases of Kuperberg spiders.

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