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Positional Number Systems with Digits Forming an Arithmetic Progression

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POSITIONAL NUMBER SYSTEMS WITH DIGITS FORMING AN ARITHMETIC PROGRESSION

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Abstract. A novel digit system that arises in a natural way in a graph-theoretical problem is studied. It is defined by a set of positive digits forming an arithmetic progression and, necessarily, a complete residue system modulo the base b. Since this is not enough to guarantee existence of a digital representation, the most significant digit is allowed to come from an extended set. We provide explicit formulæ for the jth digit in such a representation as well as for the length. Furthermore, we study digit frequencies and average lengths, thus generalising classical results for the base-b representation. For this purpose, an appropriately adapted form of the Mellin-Perron approach is employed.

1. Introduction

The concept of digital expansions is fundamental to various branches of mathematics, computer science and cryptography. We mention number-theoretic algorithms, the construction of pseudo-random sequences and the analysis of particular algorithms and data structures. For some general background we refer the reader to [2, 5, 9].

A fundamental question with respect to digital expansions is the *distribution of digits*, and (as a consequence thereof) the *sum of digits* function. A breakthrough has been achieved by H. Delange [1] in 1975, when he showed by elementary methods that for the sum of digits function $s_b(n)$ in the b-ary number system the following exact formula holds:

$$\frac{1}{N}\sum_{0\le n<N} s_b(n) = \frac{b-1}{2}\log_b N + F(\log_b N),$$

with a certain 1-periodic function $F(x)$. Subsequently, Delange's approach has been applied to many problems related to digits, see the references in the above cited papers.

A further cornerstone was published by Flajolet et al. in 1994 [2], when Delange's elementary method was replaced by the *Mellin-Perron* summation formula, which generalises Perron's summation formula for the partial sums of the coefficients of a Dirichlet series. This approach again found many followers.

A third approach to deal with problems about frequencies of digits was developed using *singular measures and exponential sums.* An early reference is [8]; the idea was subsequently applied to various other problems.

Digit expansions have been designed for bases like b^k, Fibonacci numbers (Zeckendorf expansion), $r_0r_1\dots r_k$ (Cantor expansion) and many others, including complex numbers which have to satisfy certain properties that are very challenging to discover. Examples of digits are $0,1,\dots,b-1$ (standard), $-1,0,1$ (standard ternary and redundant binary expansion), etc. It is a common feature that '0' is always a digit.

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It came as a surprise that in a graph theoretic context, two of the present writers [6] discovered a number system with base 2 and digits 1 and 4. Certain positive integers like 2 and 5 cannot be represented as $\sum_{k=0}^{\ell} a_k 2^k$, with $a_k \in \{1,4\}$, but if one allows as the leading digit a_ℓ also the number 2, each positive integer has a unique representation. What is essential here, is to have one *odd* and one *even* digit.

Increasing the maximum degree of the trees studied in the graph theoretic context led to number systems to higher bases $b \geq 2$ and sets of b nonnegative digits which are in an *arithmetic progression*: $a, a+\Delta, \ldots, a+(b-1)\Delta$. It is clear that we need one digit for each residue class modulo b. From elementary number theory it follows that the greatest common divisor of the step size Δ and the base b must be 1, which is also sufficient. As before, we must allow for a few extra digits which can only occur as leading digit. The case of the standard b-ary expansions is contained as the special case $(a,\Delta) = (0,1)$.

The obvious question that comes to mind is how the digits are distributed, among, say, the first N integers. This question will be addressed in this paper, and the Mellin-Perron technique will be used, in a slightly more general form, as given in [3]. But an even simpler question which isn't really a question for the standard b-ary system, say, must be addressed: the length of the representation! In general, the length is not a monotone function anymore, and a larger number may have a shorter representation! In the case of a complex base, as in [3], where one has *fundamental regions* with a *fractal* boundary, this is a common phenomenon.

In base b, the length of the standard representation (index of the leading digit) of n is given by $\lfloor \log_b n \rfloor$. We will find similar formulæ, but with several terms, all expressed with floor functions.

The next section contains the announcement of all our results, while their proofs are postponed to subsequent sections. It contains a formal proof of existence and uniqueness of the expansion in question, a formula for the length of the representation and a formula that gives the jth digit of the representation of n explicitly. Such formulæ are known, say, for the standard b-ary representation, where it is given in terms of floor functions. A Delange type analysis of the frequencies of digits is always based on such an explicit formula. We could have done such an analysis also in our instance, but decided to use the Mellin-Perron approach, as the computations are then less cumbersome. After the explicit formulæ for length and digits, we also study the averages and obtain asymptotic results involving periodic functions that are completely described via their Fourier coefficients. All our results are exemplied by the original system with base 2 and digits 1 and 4, but also other ones.

2. Results

We consider digit expansions to a positive integer base $b \geq 2$ with digits from $\mathcal{D} = \{a + r\Delta \mid 0 \leq r \leq b-1\}$ for some integers $a \geq 0$ and $\Delta \geq 1$ with $\gcd(b,\Delta) = 1$. Since in general, this is not sufficient to represent all positive integers, we allow the most significant digit to belong to the set $\mathcal{D}_f = \{a + r\Delta - kb \mid 0 \leq r \leq b-1, 0 \leq k < (a+r\Delta)/b\}$. Note that by definition, all elements of $\mathcal{D}_f$ are positive, even if $a = 0$.

Definition 2.1. Let n be a positive integer and $(\varepsilon_\ell, \ldots, \varepsilon_0)$ be a tuple with $\ell \geq 0$, $\varepsilon_j \in \mathcal{D}$ for $0 \leq j < \ell$ and $\varepsilon_\ell \in \mathcal{D}_f$. If

$$n = \sum_{j=0}^{\ell} \varepsilon_j b^j,$$

then $(\varepsilon_\ell, \ldots, \varepsilon_0)$ is called a (b,a,Δ)-*expansion of* n.

In the proof of existence and uniqueness, but also in the analysis of the expansion, the following notations are useful.

Definition 2.2. Let n be a positive integer and $\eta = a + r\Delta$ be the unique element of $\mathcal{D}$ with $n \equiv \eta \pmod{b}$. We set

$$\varepsilon(n) := \begin{cases} \eta, & \text{if } \eta < n, \\ n, & \text{if } \eta \geq n. \end{cases}$$

Furthermore, we set

$$T(n) := \frac{n - \varepsilon(n)}{b}.$$

Note that η is uniquely defined since we assumed that $\gcd(b, \Delta) = 1$.

Theorem 1. *Let n be a positive integer. Then n admits exactly one (b, a, Δ)-expansion.*

In particular, this expansion is given by $(\varepsilon(T^\ell(n)), \ldots, \varepsilon(T(n)), \varepsilon(n))$, where ℓ is the least non-negative integer with $T^{\ell+1}(n) = 0$.

Proof. It is obvious that $0 \le T(n) < n$ for every positive integer n. Therefore, there is an ℓ with $T^{\ell+1}(n) = 0$. Iterating $n = \varepsilon(n) + bT(n)$ yields

$$n = T^{\ell+1}(n)b^{\ell+1} + \sum_{j=0}^{\ell} \varepsilon(T^j(n))b^j = \sum_{j=0}^{\ell} \varepsilon(T^j(n))b^j.$$

By definition, $\varepsilon(n) \in \mathcal{D}$ unless $\varepsilon(n) = n$. In this case, we have $T(n) = 0$, i.e., we are considering the most significant digit, and we have $n = \eta - kb$ for some $k \ge 0$. Since $n > 0$, we conclude that $n = \varepsilon(n) \in \mathcal{D}_f$. Thus $(\varepsilon(T^\ell(n)), \ldots, \varepsilon(T(n)), \varepsilon(n))$ is indeed a (b, a, Δ)-expansion of n.

Next, we prove uniqueness. Choose the least positive integer n that admits two different (b, a, Δ)-expansions $(\varepsilon_\ell, \ldots, \varepsilon_0)$ and $(\eta_{\ell'}, \ldots, \eta_0)$, say. By minimality of n, we must have $\varepsilon_0 \ne \eta_0$, since otherwise, $(\varepsilon_\ell, \ldots, \varepsilon_1)$ and $(\eta_{\ell'}, \ldots, \eta_1)$ would be two different expansions of $(n - \varepsilon_0)/b$. Modulo b, we have $\eta_0 \equiv n \equiv \varepsilon_0 \pmod{b}$. Since $\mathcal{D}$ is a complete residue system modulo b, we conclude that at least one of ε_0 and η_0, say η_0, is not an element of $\mathcal{D}$, but is an element of $\mathcal{D}_f$. But this implies that $\ell' = 0$ and $n = \eta_0$. Since $\varepsilon_0 \ne \eta_0$, we cannot have $\ell = 0$, too. Therefore, $\eta_0 = n > \varepsilon_0$, $\varepsilon_0 \in \mathcal{D}$. This is a contradiction, since $\eta_0 \equiv \varepsilon_0 \pmod{b}$. □

Definition 2.3. Let n be a positive integer. Its unique (b, a, Δ)-expansion is denoted by $(\varepsilon_{\ell(n)}(n), \ldots, \varepsilon_0(n))$.

By Theorem 1, we obviously have $\varepsilon_0(n) = \varepsilon(n)$, i.e., $\varepsilon(n)$ is the least significant digit of the (b, a, Δ)-expansion of n.

The first part of our results concerns precise formulæ for the jth digit $\varepsilon_j(n)$ of the (b, a, Δ)-expansion of a given integer n as well as for the length of its (b, a, Δ)-expansion.

As usual, the fractional part $x - \lfloor x \rfloor$ of a real number x is denoted by $\{x\}$. We denote the inverse of b modulo $\Delta(b-1)$ by $\bar{b}$. The order of b modulo $\Delta(b-1)$ is denoted by o.

Theorem 2. *Let n be a positive integer with $n = m\Delta + d$ for some integer m and some $d \in \{0, \ldots, \Delta - 1\}$ and $j < \ell(n)$. Then we have $\varepsilon_j(n) = a + r\Delta$ for some $0 \le r \le b - 1$ if and only if*

$$\left\{ \frac{m + \lceil c \rceil}{b^{j+1}} - c \cdot \bar{b}^{j+1} \right\}$$

lies in the interval $[r/b, (r+1)/b)$, where $c = \frac{d(b-1)+a}{\Delta(b-1)}$.

This theorem is proved in Section 3.

For $(b, a, \Delta) = (2, 1, 3)$, we get the following result.

Corollary 2.4. *Let $(b, a, \Delta) = (2, 1, 3)$, n be a positive integer and $j < \ell(n)$.*

(1) *If $n \equiv 0 \pmod 3$, then $\varepsilon_j = 1$ if and only if*

$$\left\{ \frac{n+3}{3 \cdot 2^{j+1}} - \frac{1}{3}(-1)^{j+1} \right\} \in \left[0, \frac{1}{2}\right).$$

(2) *If $n \equiv 1 \pmod 3$, then $\varepsilon_j = 1$ if and only if*

$$\left\{ \frac{n+2}{3 \cdot 2^{j+1}} - \frac{2}{3}(-1)^{j+1} \right\} \in \left[0, \frac{1}{2}\right).$$

(3) *If $n \equiv 2 \pmod 3$, then $\varepsilon_j = 1$ if and only if*

$$\left\{ \frac{n+1}{3 \cdot 2^{j+1}} \right\} \in \left[0, \frac{1}{2}\right).$$

Proof. This is an immediate consequence of Theorem 2. □

The length of the (b, a, Δ)-expansion can be computed by the following explicit formula.

Theorem 3. *Let $0 \le d < \Delta$. For an integer w, let*

$$E_w = \Big\{\mu \in \mathcal{D}_f \;\Big|\; \mu \equiv \bar{b}^w d + a\frac{\bar{b}^w - 1}{b-1} \pmod{\Delta}\Big\}. \tag{2.1}$$

If E_w is non-empty, we set

$$m_w := a + (b-1)\min E_w. \tag{2.2}$$

The integers u in $\{0, \dots, o-1\}$ with $E_u \ne \emptyset$ (there is at least one such integer) are denoted by $u_1, \dots, u_t$, where $0 \le u_1 < u_2 < \dots < u_t < o$.

Let n be a positive integer with $n \equiv d \pmod{\Delta}$. Then we have

$$\ell(n) = u_t + (o + u_1 - u_t)\left\lfloor \frac{1}{o}\log_b \frac{n(b-1)+a}{b^{u_1} m_{u_1}}\right\rfloor + \sum_{i=2}^{t}(u_i - u_{i-1})\left\lfloor \frac{1}{o}\log_b \frac{n(b-1)+a}{b^{u_i} m_{u_i}}\right\rfloor. \tag{2.3}$$

We note that since $b \equiv 1 \pmod{b-1}$, we also have $\bar{b} \equiv 1 \pmod{b-1}$, thus $b-1$ divides $\bar{b}^w - 1$ and the definition of E_w in (2.1) makes sense.

This theorem is proved in Section 4.

Note that the explicit formula (2.3) heavily depends on the remainder of n modulo Δ. In some sense, we have a collection of length formulæ for the various residue classes.

As examples, we list the results for $(b, a, \Delta) \in \{(2, 1, 3), (4, 1, 3)\}$.

Corollary 2.5. *Let $(b, a, \Delta) = (2, 1, 3)$ and n be a positive integer. Then*

$$\ell(n) = \begin{cases} 2\left\lfloor \frac{1}{2}\log_2 \frac{n+1}{4}\right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3},\\ 2\left\lfloor \frac{1}{2}\log_2 \frac{n+1}{2}\right\rfloor, & \text{if } n \equiv 1 \pmod{3},\\ \left\lfloor \log_2 \frac{n+1}{3}\right\rfloor, & \text{if } n \equiv 2 \pmod{3}. \end{cases} \tag{2.4}$$

We have $\varepsilon_{\ell(n)}(n) = 2$ if and only if $n \equiv 2 \pmod{3}$.

Proof. For $n \not\equiv 2 \pmod{3}$, (2.4) is a direct consequence of (2.3). For $n \equiv 2 \pmod{3}$, (2.3) yields

$$\ell(n) = 1 + \left\lfloor \frac{1}{2}\log_2 \frac{n+1}{3}\right\rfloor + \left\lfloor \frac{1}{2}\log_2 \frac{n+1}{6}\right\rfloor.$$

This is equal to $\left\lfloor \log_2 \frac{n+1}{3}\right\rfloor$, cf. for instance [4, Eqn. (3.26)].

The additional assertion on $\varepsilon_\ell(n)$ follows from the fact that

$$n \equiv \varepsilon_\ell 2^\ell + 2^\ell - 1 \equiv (\varepsilon_\ell + 1)(-1)^\ell - 1 \pmod{3},$$

where we use the abbreviation $\ell = \ell(n)$. We therefore have $n \equiv 2 \pmod{3}$ if and only if $\varepsilon_\ell + 1$ is a multiple of 3, which is clearly equivalent to $\varepsilon_\ell = 2$. □

Corollary 2.6. *Let $(b, a, \Delta) = (4, 1, 3)$ and n be a positive integer. Then*

$$\ell(n) = \begin{cases} 2 + \left\lfloor \log_{64} \frac{3n+1}{10}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{28}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{64}\right\rfloor, & \text{if } n \equiv 0 \pmod{2},\\ 2 + \left\lfloor \log_{64} \frac{3n+1}{4}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{40}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{112}\right\rfloor, & \text{if } n \equiv 1 \pmod{2},\\ 2 + \left\lfloor \log_{64} \frac{3n+1}{7}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{16}\right\rfloor + \left\lfloor \log_{64} \frac{3n+1}{160}\right\rfloor, & \text{if } n \equiv 2 \pmod{2}. \end{cases}$$

Proof. This is a direct consequence of (2.3). □

The second part of our results are results on averages: We give a precise asymptotic description of the number of occurrences of a certain digit $\eta \in \mathcal{D}$ in the (b, a, Δ)-expansions of all positive integers up to a certain N.

We will use Iverson's notation (popularised in [4]) [*condition*] = 1 if *condition* is true and [*condition*] = 0 otherwise. We also need a notation for the *upper fractional part* $\mathsf{ufrac}\, x$ of a real number x defined by $\mathsf{ufrac}\, x := \{x\} + [x \in \mathbb{Z}] = x - (\lceil x \rceil - 1)$, i.e., $\mathsf{ufrac}\, x$ is the unique real number with $0 < \mathsf{ufrac}\, x \le 1$ and $x - \mathsf{ufrac}\, x \in \mathbb{Z}$. We will also use the Hurwitz ζ function

$$\zeta(s, \rho) = \sum_{n=0}^{\infty} \frac{1}{(n+\rho)^s}$$

for $0 < \rho \le 1$. As usual, we write $\zeta(s)$ for $\zeta(s, 1)$, the Riemann ζ function.

Theorem 4. *Let $\eta \in \mathcal{D}$ be a digit. The number of occurrences of the digit η in the (b, a, Δ)-expansion of a positive integer n is denoted by*

$$s_\eta(n) = \sum_{j=0}^{\ell(n)} [\varepsilon_j(n) = \eta]$$

and we set $s_\eta(0) = 0$.

Let

$$\alpha = \frac{a}{b-1}, \qquad \rho_1 = \mathsf{ufrac}\, \frac{\alpha+\eta}{b}, \qquad \rho_2 = \mathsf{ufrac}\, \frac{\alpha+\eta+\Delta}{b}, \tag{2.5}$$

$$f_\eta(n) = \begin{cases} s_\eta(n) - s_\eta(n-\Delta), & \text{if } n \ge \Delta, \\ 0, & \text{if } n < \Delta, \end{cases} \tag{2.6}$$

$$g_\eta(n) = f_\eta(n) - [n \equiv \eta \pmod b] + [n \equiv \eta + \Delta \pmod b], \tag{2.7}$$

$$C_{\eta,0} = \frac{\frac{1}{2} - \frac{1}{\log b}}{b} + \frac{\log_b\left(\frac{\Gamma(\rho_1)}{\Gamma(\rho_2)}\right) + \rho_1 - \rho_2 + \sum_{n=0}^{\Delta-1} s_\eta(n) - \sum_{-\alpha<n<a+b\Delta} g_\eta(n)\log_b(n+\alpha)}{\Delta}, \tag{2.8}$$

$$C_{\eta,k} = \frac{\zeta(\chi_k, \rho_1) - \zeta(\chi_k, \rho_2) + \sum_{-\alpha<n<a+b\Delta} \frac{g_\eta(n)}{(n+\alpha)^{\chi_k}}}{\Delta\chi_k(\chi_k+1)\log b} \qquad \text{with } \chi_k := \frac{2k\pi i}{\log b}, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{2.9}$$

Define the function Φ_η by its Fourier expansion

$$\Phi_\eta(x) = \sum_{k\in\mathbb{Z}} C_{\eta,k} \exp(2k\pi i x). \tag{2.10}$$

Then Φ_η is a 1-periodic continuous function and for $\delta > 0$ and positive integers N, we have

$$\sum_{n<N} s_\eta(n) = \frac{1}{b} N \log_b N + N\Phi_\eta(\log_b(N+\alpha)) + O(N^{1/2+\delta}). \tag{2.11}$$

This theorem is proved in Section 5.

Note that we do not exclude the case that n is negative in (2.6), we simply have $f_\eta(n) = 0$ for $n < 0$. We emphasise that the three sums in the definition of $C_{\eta,0}$ and $C_{\eta,k}$ are finite sums. Thus the formulæ for $C_{\eta,0}$ and $C_{\eta,k}$ are explicit, these constants only depend on b, a, Δ and η.

As an example, we consider the $(2, 1, 3)$-expansion and the digit 1.

Corollary 2.7. *For $(b, a, \Delta) = (2, 1, 3)$, we have*

$$C_{1,0} = -\frac{6 + 4\log 3 - 5\log 2 + 2\log \pi}{12 \log 2},$$

$$C_{1,k} = -\frac{i\left(\zeta\left(\frac{2k\pi i}{\log 2}\right) + 3^{-\frac{2k\pi i}{\log 2}} + 1\right)\log 2}{6k\pi(2k\pi i + \log 2)}, \qquad k \ne 0,$$

$$\Phi_1(x) = \sum_{k\in\mathbb{Z}} C_{1,k} \exp(2k\pi i x).$$

Then

$$\sum_{n<N} s_1(n) = \frac{1}{2} N \log_2 N + N\Phi_1(\log_2(N+1)) + O(N^{1/2+\delta})$$

for $\delta > 0$.

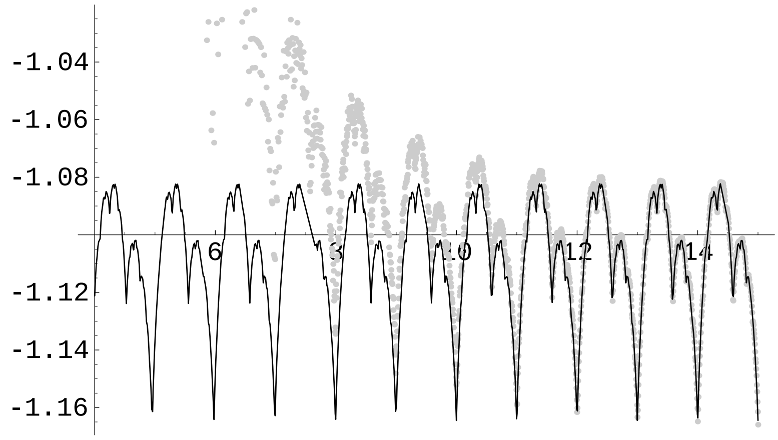

FIGURE 1. Fluctuation for $(b, a, \Delta) = (2, 1, 3)$ and $\eta = 1$: The gray dots are $(S_1(N) - 1/2N \log_2 N)/N$, the black line is $\Phi_1(x)$ (approximated with 101 Fourier coefficients). The x-axis is scaled logarithmically (base 2).

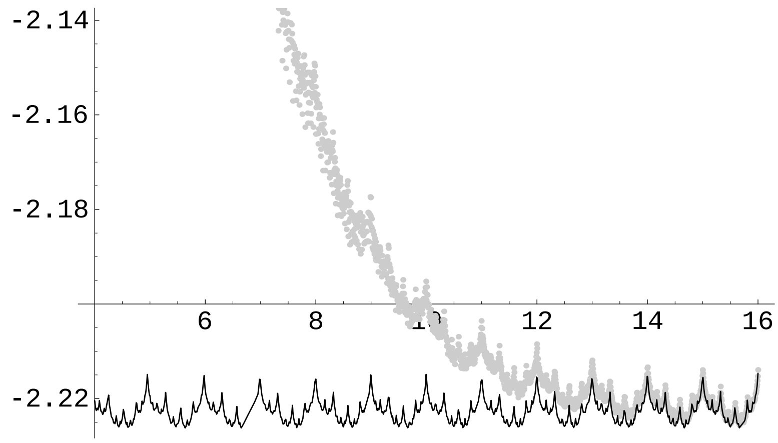

FIGURE 2. Fluctuation for $(b, a, \Delta) = (2, 1, 7)$ and $\eta = 8$: The gray dots are $(S_8(N) - 1/2N \log_2 N)/N$, the black line is $\Phi_8(x)$ (approximated with 201 Fourier coefficients). The x-axis is scaled logarithmically (base 2).

As an illustration, we compare $S_\eta(N) := \sum_{n<N} s_\eta(n)$ with the fluctuations Φ_η for some values of (b, a, Δ) and η in Figures 1, 2 and 3. Of course, the main term $(1/b)N \log_b N$ is subtracted and everything is normalised by dividing by N.

Next, we investigate the average number of occurrences of a digit $\mu \in \mathcal{D}_f$ as most significant digit of a (b, a, Δ) expansion.

Theorem 5. *Let* $\mu \in \mathcal{D}_f$ *and*

$$s_\eta^{(f)}(n) := \left[\varepsilon_{\ell(n)}(n) = \eta\right].$$

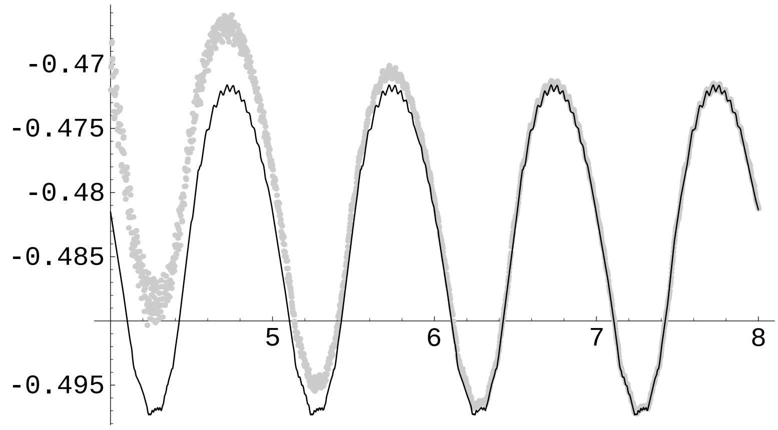

FIGURE 3. Fluctuation for $(b, a, \Delta) = (5, 7, 23)$ and $\eta = 30$: The gray dots are $(S_{30}(N) - 1/5N \log_5 N)/N$, the black line is $\Phi_{30}(x)$ (approximated with 201 Fourier coefficients). The x-axis is scaled logarithmically (base 5).

Setting

$$\alpha = \frac{a}{b-1},$$

$$\psi(x) = \frac{b^{1-\{x\}}}{b-1} + \{x\}, \tag{2.12}$$

$$\Psi_\eta(x) = \frac{\psi\big(x - \log_b(\eta+\alpha+\Delta)\big) - \psi\big(x - \log_b(\eta+\alpha)\big) + \log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha}}{\Delta}, \tag{2.13}$$

then $\Psi_\eta(x)$ is a 1-periodic continuous function. The average number of occurrences of η as leading digit of the (b, a, Δ)-expansion of the positive integers up to N equals

$$\frac{1}{N}\sum_{n=1}^{N} s_\eta^{(f)}(n) = \Psi_\eta(\log_b N) + O\left(\frac{\log N}{N}\right).$$

This theorem is proved in Section 6.

Remark. From the definition of $\Psi_\eta(x)$, it is clear that its average value equals

$$\int_0^1 \Psi_\eta(x)\,dx = \frac{1}{\Delta}\log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha},$$

since the integrals over the functions ψ cancel.

Remark. If $\Delta = (b^k - 1)(\eta + \alpha)$ for some integer k, then $\{\log_b(\eta+\alpha+\Delta)\} = \{\log_b(\eta+\alpha)\}$ and $\Psi_\eta(x) = \frac{k}{\Delta}$ is constant. In particular, this is the case for $(b, a, \Delta) = (2, 1, 3)$ and $\eta = 2$ (with $k = 1$).

Finally, we analyse the average length of (b, a, Δ)-expansions.

Theorem 6. *Let*

$$\alpha = \frac{a}{b-1}, \tag{2.14}$$

$$C_0 = \frac{1}{2} - \frac{1}{\log b} + \frac{\sum_{0\le n<\Delta}\ell(n) - \sum_{\Delta\le n<a+b\Delta}(\ell(n)-\ell(n-\Delta))\log_b(n+\alpha)}{\Delta}, \tag{2.15}$$

$$C_k = \frac{\sum_{\Delta\le n<a+b\Delta}(\ell(n)-\ell(n-\Delta))(n+\alpha)^{-\chi_k}}{\Delta\chi_k(\chi_k+1)\log b} \quad \text{with } \chi_k := \frac{2k\pi i}{\log b}, \quad k \in \mathbb{Z}\setminus\{0\}, \tag{2.16}$$

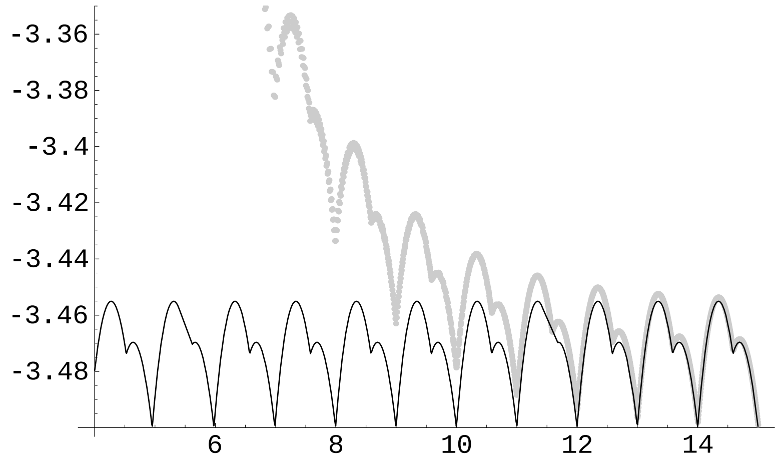

FIGURE 4. Fluctuation in the length for $(b, a, \Delta) = (2, 1, 3)$: The gray dots are $(\sum_{n<N} \ell(n) - N \log_2 N)/N$, the black line is $\Phi(x)$ (approximated with 101 Fourier coefficients). The x-axis is scaled logarithmically (base 2).

where we set $\ell(0) = -1$, *and define the function* Φ *by its Fourier expansion*

$$\Phi(x) := \sum_{k\in\mathbb{Z}} C_k \exp(2k\pi i x).$$

Then Φ *is a* 1*-periodic continuous function and for positive integers* N, *we have*

$$\sum_{n<N} \ell(n) = N \log_b(N) + N\Phi(\log_b(N+\alpha)) + O(\log N).$$

This theorem is proved in Section 7.

Again, we consider the example $(b, a, \Delta) = (2, 1, 3)$.

Corollary 2.8. *Let* $(b, a, \Delta) = (2, 1, 3)$,

$$\begin{aligned} C_0 &= -\frac{6 + 9\log 2 + 2\log 3}{6\log 2}, \\ C_k &= -\frac{3^{-\frac{2k\pi i}{\log 2}}\log 2 + \log 4}{12k^2\pi^2 - 6k\pi i \log 2}, \qquad k \neq 0, \\ \Phi(x) &= \sum_{k\in\mathbb{Z}} C_k \exp(2k\pi i x). \end{aligned}$$

Then we have

$$\sum_{n<N} \ell(n) = N \log_2(N) + N\Phi(\log_2(N+1)) + O(\log N).$$

This is illustrated in Figure 4.

It is even possible to obtain explicit formulæ for the average length of an expansion by means of the expression for $\ell(n)$ given in Theorem 3. However, the resulting formulæ are rather lengthy, and so we only show this for the particular example $(b, a, \Delta) = (2, 1, 3)$, where formulæ for $\ell(n)$ are given in Corollary 2.5.

Theorem 7. *For* (b, a, Δ), *we have*

$$\begin{aligned} \sum_{n=1}^{N} \ell(n) = {} & 2\left(\left\lfloor \tfrac{N}{3} \right\rfloor + \tfrac{4}{3}\right)\left\lfloor \tfrac{1}{2}\log_2 \tfrac{N+1}{4} \right\rfloor + 2\left(\left\lfloor \tfrac{N+2}{3} \right\rfloor + \tfrac{2}{3}\right)\left\lfloor \tfrac{1}{2}\log_2 \tfrac{N+1}{2} \right\rfloor + \left\lfloor \tfrac{N+4}{3} \right\rfloor \left\lfloor \log_2 \tfrac{N+1}{3} \right\rfloor + \left\lfloor \tfrac{N}{3} \right\rfloor \\ & - \tfrac{8}{9}(N+1)\cdot\left(4^{-\{\log_4(N+1)\}} + 4^{-\{\log_4 \frac{N+1}{2}\}} + 3\cdot 2^{-2-\{\log_2 \frac{N+1}{3}\}}\right) + \tfrac{22}{3}. \end{aligned} \tag{2.17}$$

This theorem is proved in Section 8.

3. Digit formula — Proof of Theorem 2

This section is devoted to the proof of the digit formula enunciated in Theorem 2.

We write $\varepsilon_k(n) = a + r_k\Delta$ and $\ell = \ell(n)$. Then we have

$$n - a \cdot \frac{b^{j+1}-1}{b-1} = \Delta \sum_{k=0}^{j} r_k b^k + \sum_{k=j+1}^{\ell} \varepsilon_k(n) b^k.$$

Now set $q = \left(a \cdot \frac{b^{j+1}-1}{b-1} - d\right) \bar{b}^{j+1}$, and consider the expression

$$M = n - a \cdot \frac{b^{j+1}-1}{b-1} + q \cdot b^{j+1} = \Delta \sum_{k=0}^{j} r_k b^k + q \cdot b^{j+1} + \sum_{k=j+1}^{\ell} \varepsilon_k(n) b^k.$$

By our choice of q and $\bar{b}$, this number is congruent to

$$d - a \cdot \frac{b^{j+1}-1}{b-1} + \left(a \cdot \frac{b^{j+1}-1}{b-1} - d\right) \bar{b}^{j+1} b^{j+1} = 0$$

modulo Δ. Hence

$$\frac{M}{\Delta} = \sum_{k=0}^{j} r_k b^k + \frac{q \cdot b^{j+1} + \sum_{k=j+1}^{\ell} \varepsilon_k(n) b^k}{\Delta}$$

is an integer. Note also that the second summand is divisible by b^{j+1}, as $\gcd(\Delta, b) = 1$. If follows that

$$\left\{\frac{M}{\Delta b^{j+1}}\right\} = \left\{\sum_{k=0}^{j} r_k b^{k-j-1}\right\} = \frac{r_j}{b} + \sum_{k=0}^{j-1} r_k b^{k-j-1}.$$

The second summand is trivially estimated to lie between 0 and $\frac{1}{b} - b^{-j-1}$. Thus, in order to prove our theorem, it is sufficient to prove the inequality

$$0 \le \left\{\frac{m + \lceil c \rceil}{b^{j+1}} - c \cdot \bar{b}^{j+1}\right\} - \left\{\frac{M}{\Delta b^{j+1}}\right\} < b^{-j-1},$$

which then implies that

$$\frac{r_j}{b} \le \left\{\frac{m + \lceil c \rceil}{b^{j+1}} - c \cdot \bar{b}^{j+1}\right\} < \frac{r_j + 1}{b},$$

as claimed. So let us consider the difference

$$\begin{aligned}
\frac{m + \lceil c \rceil}{b^{j+1}} &- c \cdot \bar{b}^{j+1} - \frac{M}{\Delta b^{j+1}} \\
&= \frac{(m + \lceil c \rceil)\Delta(b-1) - (d(b-1)+a)b^{j+1}\bar{b}^{j+1} - \left(n - a \cdot \frac{b^{j+1}-1}{b-1} + q \cdot b^{j+1}\right)(b-1)}{\Delta(b-1)b^{j+1}} \\
&= \frac{(m + \lceil c \rceil)\Delta(b-1) - (d(b-1)+a)(b\bar{b})^{j+1} - (\Delta m + d + q b^{j+1})(b-1) + a(b^{j+1}-1)}{\Delta(b-1)b^{j+1}} \\
&= \frac{(\lceil c \rceil\Delta - d)(b-1) - a - (d(b-1)+a)(b\bar{b})^{j+1} - (b-1)q b^{j+1} + a b^{j+1}}{\Delta(b-1)b^{j+1}} \\
&= \frac{(\lceil c \rceil\Delta - d)(b-1) - a}{\Delta(b-1)b^{j+1}} - \frac{(d(b-1)+a)\bar{b}^{j+1} + (b-1)q - a}{\Delta(b-1)} \\
&= \frac{(\lceil c \rceil\Delta - d)(b-1) - a}{\Delta(b-1)b^{j+1}} - \frac{(d(b-1)+a)\bar{b}^{j+1} + (b-1)\left(a \cdot \frac{b^{j+1}-1}{b-1} - d\right)\bar{b}^{j+1} - a}{\Delta(b-1)} \\
&= \frac{(\lceil c \rceil\Delta - d)(b-1) - a}{\Delta(b-1)b^{j+1}} - \frac{a((b\bar{b})^{j+1} - 1)}{\Delta(b-1)}.
\end{aligned}$$

The second fraction is an integer by our choice of $\bar{b}$. Now note that the numerator of the first fraction can also be written as

$$(\lceil c\rceil\Delta - d)(b-1) - a = \Delta(b-1)\left(\left\lceil\frac{d(b-1)+a}{\Delta(b-1)}\right\rceil - \frac{d(b-1)+a}{\Delta(b-1)}\right),$$

which shows that it lies between 0 and $\Delta(b-1)-1$. Equation (3) readily follows, since we know that $\frac{M}{\Delta b^{j+1}}$ is a fraction with denominator b^{j+1}.

4. Length Formula — Proof of Theorem 3

Before proving Theorem 3, we prove a lemma which characterises the length and the most significant digit of the (b, a, Δ)-expansion of an integer.

Lemma 4.1. *Let n be a positive integer, $\ell \geq 0$ and $\mu \in \mathcal{D}_f$. Then the following two conditions are equivalent.*

(1) *$\ell(n) = \ell$ and $\varepsilon_\ell(n) = \eta$.*
(2) *We have*

$$\begin{gathered} n(b-1)+a \equiv b^\ell(\eta(b-1)+a) \pmod{\Delta(b-1)}, \\ 0 \leq \frac{n(b-1)+a}{\Delta(b-1)b^\ell} - \frac{\eta(b-1)+a}{\Delta(b-1)} < 1. \end{gathered} \tag{4.1}$$

Proof. We have $\ell(n) = \ell$ and $\varepsilon_\ell(n) = \eta$ if and only if there are integers $r_j \in \{0, \dots, b-1\}$ such that

$$n = \mu b^\ell + a\frac{b^\ell - 1}{b-1} + \Delta\sum_{j=0}^{\ell-1} r_j b^j.$$

Set $x = \sum_{j=0}^{\ell-1} r_j b^j$. The conditions on the r_j are equivalent to saying that x is an integer with $0 \leq x < b^\ell$. Thus the first condition of the lemma is equivalent to

$$n = \mu b^\ell + a\frac{b^\ell - 1}{b-1} + \Delta x, \qquad x \in \mathbb{Z}, \qquad 0 \leq x < b^\ell. \tag{4.2}$$

Multiplying this by $(b-1)$ and rearranging shows that (4.2) is equivalent to

$$n(b-1) + a = b^\ell(\eta(b-1)+a) + \Delta(b-1)x, \qquad x \in \mathbb{Z}, \qquad 0 \leq x < b^\ell. \tag{4.3}$$

It is easily seen that (4.3) is equivalent to (4.1). □

The description of $\ell(n)$ in Lemma 4.1 can be used to obtain partial information on $\ell(n)$ in the form of an explicit formula for $\lfloor(\ell(n)-u)/o\rfloor$ for certain $u \in \{0, \dots, o-1\}$.

Lemma 4.2. *Let $0 \leq d < \Delta$. Choose $u \in \{0, \dots, o-1\}$ such that E_u as defined in (2.1) is non-empty. Let n be a positive integer with $n \equiv d \pmod{\Delta}$.*

Then we have

$$\left\lfloor\frac{\ell(n)-u}{o}\right\rfloor = \left\lfloor\frac{1}{o}\log_b\frac{n(b-1)+a}{b^u m_u}\right\rfloor \tag{4.4}$$

where m_u is defined in (2.2).

Proof. We use the abbreviations $\ell = \ell(n)$ and $\varepsilon_\ell = \varepsilon_\ell(n)$. We write $\ell = Lo + u + v$ for some integer L and some $v \in \{0, \dots, o-1\}$. Obviously, this implies $L = \lfloor(\ell-u)/o\rfloor$.

Since

$$m_u \equiv \bar{b}^u\big((b-1)d + a\big) \pmod{(b-1)\Delta}$$

by (2.1) and (2.2) and

$$(\varepsilon_\ell(b-1)+a)b^v \equiv \bar{b}^{u+Lo}(n(b-1)+a) \equiv \bar{b}^u\big((b-1)d+a\big) \pmod{(b-1)\Delta}$$

by (4.1), we have

$$m_u \equiv (\varepsilon_\ell(b-1)+a)b^v \pmod{(b-1)\Delta}. \tag{4.5}$$

By construction, all elements of D_f are positive. In particular $\min E_u > 0$ and $\varepsilon_\ell > 0$.

From (4.1), we obtain

$$\frac{\varepsilon_\ell(b-1)+a}{m_u} b^v \cdot b^{Lo} \le \frac{n(b-1)+a}{b^u m_u} < \frac{\Delta(b-1)+\varepsilon_\ell(b-1)+a}{m_u} b^v \cdot b^{Lo}. \tag{4.6}$$

We first claim that

$$m_u \le (\varepsilon_\ell(b-1)+a)b^v. \tag{4.7}$$

Assume the contrary. Then by (4.5), there is a positive integer f such that

$$m_u = (\varepsilon_\ell(b-1)+a)b^v + f(b-1)\Delta. \tag{4.8}$$

If $v=0$, this is a contradiction to the definition of m_u. Therefore, we only have to consider the case $v>0$. We write $\min E_u = a + r\Delta - kb$ for some $0 \le r < b$ and $0 \le k < (a+\Delta r)/b$. Using these estimates and (4.8) yields

$$ab + f(b-1)\Delta < (\varepsilon_\ell(b-1)+a)b^v + f(b-1)\Delta = a + (b-1)(a+r\Delta-kb) \le ab + r(b-1)\Delta.$$

We conclude that $0 < f < r < b$. Taking (4.8) modulo b, we get

$$a - (a+r\Delta) \equiv -f\Delta \pmod{b},$$

which implies that $r \equiv f \pmod{b}$, which is a contradiction to $0 < f < r < b$ and concludes the proof of (4.7).

Next, we claim that

$$\Delta(b-1) + \varepsilon_\ell(b-1) + a \le b^{o-v} m_u. \tag{4.9}$$

By (4.5), we have $\Delta(b-1)+\varepsilon_\ell(b-1)+a \equiv b^{o-v}m_u \pmod{(b-1)\Delta}$. Therefore, there is an integer f' such that

$$\Delta(b-1) + \varepsilon_\ell(b-1) + a = b^{o-v}m_u + f'(b-1)\Delta. \tag{4.10}$$

We assume that f' is positive. We write $\varepsilon_\ell = a + \Delta r' - k'b$ and get

$$\begin{aligned} f'(b-1)\Delta + ab &< f'(b-1)\Delta + b^{o-v}(a + (b-1)\min E_u) \\ &= \Delta(b-1) + (a+\Delta r' - k'b)(b-1) + a \le (r'+1)(b-1)\Delta + ab, \end{aligned}$$

which implies $0 < f' < r'+1 \le b$. Taking (4.10) modulo b yields

$$-\Delta - (a+\Delta r') + a \equiv -f'\Delta \pmod{b},$$

i.e., $f' \equiv (r'+1) \pmod{b}$. This is a contradiction to $0 < f' < r'+1 \le b$ and therefore proves (4.9).

Combining (4.6), (4.7), and (4.9) yields

$$b^{Lo} \le \frac{n(b-1)+a}{b^u m_u} < b^{(L+1)o},$$

which immediately gives (4.4). □

The partial information obtained in Lemma 4.2 is now combined for several u and enables us to prove Theorem 3.

Proof of Theorem 3. Again, we use the abbreviation $\ell = \ell(n)$.

We write $\ell = L \cdot o + R$ for a suitable integer L and some $R \in \{0, \dots, o-1\}$. By (4.1) and (2.1), we have $\varepsilon_\ell(n) \in E_R$. In particular, $E_R \ne \emptyset$ and there is a $1 \le j \le t$ with $R = u_j$.

We may rewrite ℓ as

$$\ell = L \cdot o + u_1 + \sum_{k=2}^{t} [R \ge u_k](u_k - u_{k-1}).$$

Writing

$$L_k := \left\lfloor \frac{\ell - u_k}{o} \right\rfloor$$

for $1 \le k \le t$, we note that $L_1 = L$ and

$$[R \ge u_k] = 1 + L_k - L_1.$$

Thus

$$\ell = L_1 \cdot o + u_1 + \sum_{k=2}^{t}(1 + L_k - L_1)(u_k - u_{k-1}).$$

Collecting all coefficients of L_1, we obtain

$$\ell = u_t + L_1(o - u_t + u_1) + \sum_{k=2}^{t} L_k(u_k - u_{k-1}). \tag{4.11}$$

Plugging (4.4) in (4.11) yields (2.3). □

5. Digit Frequencies — Proof of Theorem 4

This section is devoted to the proof of Theorem 4.

We first rewrite $S_\eta(n) = \sum_{n<N} s_\eta(n)$ by summation by parts to a sum involving $(N-n)f_\eta(n)$, where $f_\eta(n)$ is defined in (2.6).

We have

$$\begin{aligned}\sum_{n<N}(N-n)f_\eta(n) &= \sum_{0\le n<N}\Delta s_\eta(n) + \sum_{N-\Delta\le n<N}(N-n-\Delta)s_\eta(n) - \sum_{0\le n<\Delta}(N-n)s_{\eta(n)}\\ &= \Delta S_\eta(N) - N\sum_{n=0}^{\Delta-1}s_\eta(n) + O(\log N).\end{aligned} \tag{5.1}$$

The remaining sum over $0 \le n < \Delta$ is a finite sum and will finally be part of $C_{\eta,0}$. Thus it is sufficient to study $\sum_{n<N}(N-n)f_\eta(n)$.

We intend to use the Mellin-Perron summation formula in the version

$$\sum_{\substack{n\\ -\alpha<n<N}}(N-n)f(n) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\Bigg(\sum_{\substack{n\\ n>-\alpha}}^{\infty}\frac{f(n)}{(n+\alpha)^s}\Bigg)(N+\alpha)^{s+1}\frac{ds}{s(s+1)}, \tag{5.2}$$

where $\alpha \in \mathbb{R}$ and C is in the half-plane of absolute convergence of the Dirichlet series $\sum_{n>-\alpha}\frac{f(n)}{(n+\alpha)^s}$, cf. [3, (4.3)]. Note that we do not impose the frequently used restriction $0 < \alpha \le 1$, but we sum over $n > -\alpha$, which amounts to the same. This version, however, smoothes the following calculations. In our application, α is positive and $f(n) = f_\eta(n)$ vanishes for negative n.

It turns out that α as defined in (2.5) is a useful choice for our problem.

Therefore, we study the Dirichlet generating function

$$\Lambda_\eta(s) := \sum_{n>-\alpha}^{\infty}\frac{f_\eta(n)}{(n+\alpha)^s}.$$

Since we trivially have $|f_\eta(n)| \le \ell(n) + \ell(n-\Delta) = O(\log n)$, this Dirichlet series converges absolutely for $\Re s > 1$, so we may choose $C = 2$ in (5.2). To get an explicit formula for $\Lambda_\eta(s)$, we first derive recursive formulæ for $s_\eta(n)$ and $f_\eta(n)$.

We obviously have

$$s_\eta(n) = \begin{cases}[\varepsilon(n) = \eta] + s_\eta(T(n)), & \text{if } n > 0,\\ 0, & \text{if } n = 0.\end{cases} \tag{5.3}$$

Assume that $n \ge a + b\Delta$ and choose $r \in \{0, \dots, b-1\}$ such that $\varepsilon(n-\Delta) = a + r\Delta$. Then $n \equiv a + (r+1)\Delta \pmod b$, which implies

$$\begin{aligned}\varepsilon(n) &= \begin{cases}a + (r+1)\Delta, & \text{if } r < b-1,\\ a, & \text{if } r = b-1\end{cases}\\ &= \varepsilon(n-\Delta) + \Delta - b\Delta\,[n \equiv a \pmod b],\end{aligned}$$

since $r = b-1$ is obviously equivalent to $n \equiv a \pmod b$. For the transformation T, this yields

$$T(n-\Delta) = \frac{n - \Delta - \varepsilon(n) + \Delta - b\Delta\,[n \equiv a \pmod b]}{b} = T(n) - \Delta\,[n \equiv a \pmod b]. \tag{5.4}$$

Plugging (5.4) and (5.3) in (2.6) yields

$$\begin{aligned} f_\eta(n) &= [\varepsilon(n)=\eta] - [\varepsilon(n-\Delta)=\eta] + s_\eta(T(n)) - s_\eta(T(n-\Delta)) \\ &= [n \equiv \eta \pmod b] - [n \equiv \eta+\Delta \pmod b] + f_\eta(T(n))\,[n \equiv a \pmod b] \\ &= [n \equiv \eta \pmod b] - [n \equiv \eta+\Delta \pmod b] + f_\eta\left(\frac{n-a}{b}\right)[n \equiv a \pmod b], \end{aligned} \tag{5.5}$$

for $n \geq a + b\Delta$, since $n \equiv a \pmod b$ is equivalent to $\varepsilon(n) = a$.

To avoid dealing with various special summands arising from the fact that in general, (5.5) is not true for $n < a+b\Delta$, we use the function $g_\eta(n)$ defined in (2.7) which takes care of these special cases. This function $g_\eta(n)$ has been defined in such a way that

$$\begin{aligned} f_\eta(n) = [n \equiv \eta \pmod b] &- [n \equiv \eta+\Delta \pmod b] \\ &+ f_\eta\left(\frac{n-a}{b}\right)[n \equiv a \pmod b] + g_\eta(n)\,[n < a+b\Delta] \end{aligned} \tag{5.6}$$

now holds for all $n \in \mathbb{Z}$, because we have $(n-a)/b < \Delta$ for $n < a + b\Delta$ and f_η vanishes for arguments less than Δ by definition.

Dividing (5.6) by $(n+\alpha)^s$ and summing over all integers $n > -\alpha$ yields the relation

$$\Lambda_\eta(s) = \sum_{\substack{n>-\alpha \\ n\equiv\eta}} \frac{1}{(n+\alpha)^s} - \sum_{\substack{n>-\alpha \\ n\equiv\eta+\Delta}} \frac{1}{(n+\alpha)^s} + \sum_{\substack{n>-\alpha \\ n\equiv a}} \frac{f_\eta((n-a)/b)}{(n+\alpha)^s} + G_\eta(s), \tag{5.7}$$

where all congruences are modulo b and

$$G_\eta(s) = \sum_{-\alpha<n<a+b\Delta} \frac{g_\eta(n)}{(n+\alpha)^s}.$$

Replacing n in the first three sums of (5.7) by $\eta + nb$, $\eta + \Delta + nb$, and $a + nb$, respectively, yields

$$\Lambda_\eta(s) = b^{-s}\zeta(s,\rho_1) - b^{-s}\zeta(s,\rho_2) + b^{-s}\Lambda_\eta(s) + G_\eta(s). \tag{5.8}$$

For the third sum, the relation $\alpha = (a+\alpha)/b$ has been used which is equivalent to (and motivates) the definition of α in (2.5). From (5.8) we immediately get the explicit expression

$$\Lambda_\eta(s) = \frac{b^{-s}\zeta(s,\rho_1) - b^{-s}\zeta(s,\rho_2) + G_\eta(s)}{1-b^{-s}}.$$

The function $\Lambda_\eta(s)$ is a meromorphic function in $\mathbb{C}$ with poles in χ_k for $k \in \mathbb{Z}$, since the poles of the ζ functions at $s = 1$ cancel. For some fixed $0 < \delta < 1/2$, we have $|\zeta(s,\rho)| \leq (\Im s)^{1-\delta}$ for $\Re s \geq -1/2+\delta$ (cf. [10, § 13.51]). Thus we can shift the line of integration in (5.2) to $\Re s = -1/2+\delta$ for some $0 < \delta < \frac{1}{2}$, where the residues in χ_k have to be taken into account. As usual, paths with $\Im s = (2M+1)\pi i/\log b$ are chosen for this purpose, where the denominator is bounded away from zero.

We obtain

$$\begin{aligned} &\sum_{-\alpha\leq n<N} (N-n) f_\eta(n) \\ &\quad= \sum_{k\in\mathbb{Z}} \operatorname{Res}\left(\Lambda_\eta(s)\frac{(N+\alpha)^{s+1}}{s(s+1)}, s=\chi_k\right) + \frac{1}{2\pi i}\int_{-\frac12+\delta-i\infty}^{-\frac12+\delta+i\infty} \Lambda_\eta(s)(N+\alpha)^{s+1}\frac{ds}{s(s+1)}. \end{aligned} \tag{5.9}$$

The integral is bounded by $O(N^{1/2+\delta})$. For $k \neq 0$, the residue at χ_k is

$$\operatorname{Res}\left(\Lambda_\eta(s)\frac{(N+\alpha)^{s+1}}{s(s+1)}, s=\chi_k\right) = \Delta C_{\eta,k}(N+\alpha)\exp(2k\pi i \log_b(N+\alpha)), \tag{5.10}$$

where $C_{\eta,k}$ is defined in (2.9). We note that $C_{\eta,k} = O(k^{-3/2}\log k)$, cf. [10, § 13.51]. At $s = 0$, the integrand has a double pole with residue

$$\operatorname{Res}\left(\Lambda_\eta(s)\frac{(N+\alpha)^{s+1}}{s(s+1)}, s=0\right) = (G_\eta(0) - \rho_1 + \rho_2)(N+\alpha)\log_b(N+\alpha) + \tilde{C}_{\eta,0}(N+\alpha), \tag{5.11}$$

where

$$\tilde{C}_{\eta,0} = \log_b \frac{\Gamma(\rho_1)}{\Gamma(\rho_2)} + \left(\frac{1}{2} + \frac{1}{\log b}\right)(\rho_1 - \rho_2) + \left(\frac{1}{2} - \frac{1}{\log b}\right) G_\eta(0) + \frac{G'_\eta(0)}{\log b}. \tag{5.12}$$

Combining (5.9), (5.10) and (5.11) yields

$$\sum_{-\alpha \le n < N} (N-n) f_\eta(n) = (G_\eta(0) - \rho_1 + \rho_2)\, N \log_b N + \tilde{C}_{\eta,0} N + N\Delta \sum_{k \in \mathbb{Z}\setminus\{0\}} C_{\eta,k} \exp(2k\pi i \log_b(N+\alpha)) + O(N^{1/2+\delta}). \tag{5.13}$$

We want to calculate the value of

$$\begin{aligned} G_\eta(0) &= \sum_{-\alpha < n < a+b\Delta} g_\eta(n) \\ &= \sum_{-\alpha < n < a+b\Delta} f_\eta(n) - \sum_{-\alpha < n < a+b\Delta} [n \equiv \eta \pmod b] + \sum_{-\alpha < n < a+b\Delta} [n \equiv \eta + \Delta \pmod b]. \end{aligned}$$

By the definition of f_η and using (5.3), the first summand can be rewritten as

$$\begin{aligned} &\sum_{\Delta \le n < a+b\Delta} (s_\eta(n) - s_\eta(n-\Delta)) \\ &\quad= \sum_{\Delta \le n < a+b\Delta} s_\eta(n) - \sum_{0 \le n < a+(b-1)\Delta} s_\eta(n) \\ &\quad= \sum_{a+(b-1)\Delta \le n < a+b\Delta} s_\eta(n) - \sum_{0 \le n < \Delta} s_\eta(n) \\ &\quad= \sum_{a+(b-1)\Delta \le n < a+b\Delta} [n \equiv \eta \pmod b] + \sum_{a+(b-1)\Delta \le n < a+b\Delta} s_\eta(T(n)) - \sum_{0 \le n < \Delta} s_\eta(n). \end{aligned}$$

Here, the second and third summand cancel: note that $0 \le T(n) \le \frac{n-a}{b} < \Delta$ for all $a+(b-1)\Delta \le n < a+b\Delta$, and that $T(n_1) \ne T(n_2)$ for distinct n_1, n_2 in this range (otherwise, $n_1 \equiv n_2 \pmod \Delta$, which is impossible). Therefore, the second sum is just a permutation of the third. It remains to compute

$$\sum_{a+(b-1)\Delta \le n < a+b\Delta} [n \equiv \eta \pmod b] = \sum_{a+(b-1)\Delta \le \eta + mb < a+b\Delta} 1 = \left\lceil \frac{a-\eta}{b} \right\rceil - \left\lceil \frac{a-\eta-\Delta}{b} \right\rceil.$$

Likewise, we have

$$\sum_{-\alpha < n < a+b\Delta} [n \equiv \eta \pmod b] = \left\lceil \frac{a-\eta}{b} \right\rceil + \left\lceil \frac{\alpha+\eta}{b} \right\rceil + \Delta - 1$$

and

$$\sum_{-\alpha < n < a+b\Delta} [n \equiv \eta + \Delta \pmod b] = \left\lceil \frac{a-\eta-\Delta}{b} \right\rceil + \left\lceil \frac{\alpha+\eta+\Delta}{b} \right\rceil + \Delta - 1.$$

Summing up, we obtain

$$G_\eta(0) = \left\lceil \frac{\alpha+\eta+\Delta}{b} \right\rceil - \left\lceil \frac{\alpha+\eta}{b} \right\rceil.$$

Since $\rho_1 = 1 + \frac{\alpha+\eta}{b} - \left\lceil \frac{\alpha+\eta}{b} \right\rceil$ and $\rho_2 = 1 + \frac{\alpha+\eta+\Delta}{b} - \left\lceil \frac{\alpha+\eta+\Delta}{b} \right\rceil$, it finally follows that

$$G_\eta(0) - \rho_1 + \rho_2 = \frac{\Delta}{b}, \tag{5.14}$$

as expected.

Inserting (5.14) in (5.12), we see that

$$\tilde{C}_{\eta,0} + \sum_{n<\Delta} s_\eta(n) = \Delta C_{\eta,0}, \tag{5.15}$$

where $C_{\eta,0}$ has been defined in (2.8).

Combining (5.1), (5.13), (5.14), (5.15) and (2.10) yields (2.11).

Since $C_{\eta,k} = O(k^{-3/2}\log k)$, the Fourier series (2.10) converges uniformly, which immediately implies that Φ_η is a continuous function. The periodicity of Φ_η is an immediate consequence of the definition as a Fourier series. This concludes the proof of Theorem 4.

Remark. The calculation of the coefficient of $N\log_b N$ in (5.13) was slightly tedious, which is a bit embarrassing in view of the intuitively "obvious" result of Δ/b. Alternatively, one could find out the value of $G_\eta(0) - \rho_1 + \rho_2$ by comparing (5.13) with some rougher estimates. However, this is not painless, neither, since the special treatment of the most significant digits prohibits a completely straightforward exactification of the "obvious" result.

6. Frequencies of Leading Digits — Proof of Theorem 5

This section is devoted to the proof of Theorem 5.

The distribution of the exceptional leading digit can be analysed by means of the Mellin-Perron approach as well. However, it is also possible to obtain the asymptotic behaviour in a more elementary way. Note that the leading digit of n equals η if and only if n can be written in the form

$$n = \eta b^\ell + a\cdot\frac{b^\ell-1}{b-1} + \Delta k$$

for some $0 \le k < b^\ell$. Thus, given the length ℓ, there are

$$\min\left(1 + \left\lfloor\frac{N - a\cdot\frac{b^\ell-1}{b-1} - \eta b^\ell}{\Delta}\right\rfloor, b^\ell\right)$$

numbers $\le N$ with leading digit η as long as this is nonnegative. Now, we have

$$1 + \left\lfloor\frac{N - a\cdot\frac{b^\ell-1}{b-1} - \eta b^\ell}{\Delta}\right\rfloor \ge b^\ell$$

for

$$\ell \le L(N) := \left\lfloor\log_b\frac{N+\alpha+\Delta}{\eta+\alpha+\Delta}\right\rfloor$$

and we have

$$1 + \left\lfloor\frac{N - a\cdot\frac{b^\ell-1}{b-1} - \eta b^\ell}{\Delta}\right\rfloor > 0$$

for

$$\ell \le M(N) := \left\lfloor\log_b\frac{N+\alpha}{\eta+\alpha}\right\rfloor.$$

Hence, the total number of occurrences of the leading digit η among all positive integers $\le N$ is given by

$$\begin{aligned}
&\sum_{\ell=0}^{L(N)} b^\ell + \sum_{\ell=L(N)+1}^{M(N)}\left(1 + \left\lfloor\frac{N - a\cdot\frac{b^\ell-1}{b-1} - \eta b^\ell}{\Delta}\right\rfloor\right)\\
&\quad= \frac{b^{L(N)+1}-1}{b-1} + \sum_{\ell=L(N)+1}^{M(N)}\frac{N-(\eta+\alpha)\,b^\ell}{\Delta} + O(\log N)\\
&\quad= \frac{b^{L(N)+1}}{b-1} + \frac{\eta+\alpha}{\Delta(b-1)}\left(b^{L(N)+1} - b^{M(N)+1}\right) + \frac{N}{\Delta}(M(N)-L(N)) + O(\log N).
\end{aligned}$$

Now write $L(N)$ and $M(N)$ as

$$\begin{aligned}
L(N) &= \log_b\frac{N+\alpha+\Delta}{\eta+\alpha+\Delta} - \left\{\log_b\frac{N+\alpha+\Delta}{\eta+\alpha+\Delta}\right\}\\
&= \log_b N - \log_b(\eta+\alpha+\Delta) - F_1(N) + O(N^{-1})
\end{aligned}$$

and

$$\begin{aligned} M(N) &= \log_b \frac{N+\alpha}{\eta+\alpha} - \left\{ \log_b \frac{N+\alpha}{\eta+\alpha} \right\} \\ &= \log_b N - \log_b (\eta+\alpha) - F_2(N) + O(N^{-1}), \end{aligned}$$

where the abbreviations $F_1(N) = \left\{ \log_b \frac{N+\alpha+\Delta}{\eta+\alpha+\Delta} \right\}$ and $F_2(N) = \left\{ \log_b \frac{N+\alpha}{\eta+\alpha} \right\}$ are used. Then we can express the average number of occurrences of η as first digit among the positive numbers $\leq N$ in the form

$$\begin{aligned} &\frac{1}{N}\left(\frac{\eta+\alpha+\Delta}{\Delta(b-1)} b^{L(N)+1} - \frac{\eta+\alpha}{\Delta(b-1)} b^{M(N)+1} + \frac{N}{\Delta}(M(N)-L(N)) + O(\log N) \right) \\ &= \frac{b}{\Delta(b-1)} \left(b^{-F_1(N)} - b^{-F_2(N)} \right) + \frac{1}{\Delta} \left(\log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha} + F_1(N) - F_2(N) \right) + O(N^{-1}\log N). \end{aligned} \tag{6.1}$$

Now note that the 1-periodic function $\psi(x)$ defined in (2.12) satisfies $\psi(0) = \lim_{x\to 1^-} \psi(x) = \frac{b}{b-1}$, so it is continuous. It is differentiable in every non-integer point; at an integer point, left and right derivatives exist (but are not equal). Thus ψ is Lipschitz-continuous. Hence we can rewrite (6.1) in yet another form:

$$\begin{aligned} &\frac{1}{\Delta}\left(\psi(F_1(N)) - \psi(F_2(N)) + \log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha} \right) + O(N^{-1}\log N) \\ &\quad = \frac{1}{\Delta}\left(\psi\left(\log_b \frac{N+\alpha+\Delta}{\eta+\alpha+\Delta} \right) - \psi\left(\log_b \frac{N+\alpha}{\eta+\alpha} \right) + \log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha} \right) + O(N^{-1}\log N) \\ &\quad = \frac{1}{\Delta}\Bigg(\psi\left(\log_b N - \log_b(\eta+\alpha+\Delta)\right) - \psi\left(\log_b N - \log_b(\eta+\alpha)\right) \\ &\qquad + \log_b \frac{\eta+\alpha+\Delta}{\eta+\alpha} \Bigg) + O(N^{-1}\log N) \\ &\quad = \Psi_\eta(\log_b N) + O(N^{-1}\log N), \end{aligned}$$

where Ψ_η is defined in (2.13). By the above noted properties of ψ, the function Ψ_η is also continuous and 1-periodic. This completes the proof of Theorem 5.

7. Average Length — Proof of Theorem 6

This section is devoted to the proof of Theorem 6.

The proof runs along the lines of the proof of Theorem 4, but the technical details are slightly simpler, so we only point out the modifications. The difference function $f_\eta(n)$ is replaced with

$$f(n) = \begin{cases} \ell(n) - \ell(n-\Delta), & \text{if } n \geq \Delta, \\ 0, & \text{if } n < \Delta. \end{cases}$$

We have the recursion

$$\ell(n) = 1 + \ell(T(n)) \qquad \text{for } n > 0,$$

which implies that all subsequent conditional expressions depending on η vanish. For instance, we simply have

$$f(n) = f\left(\frac{n-a}{b}\right)[n \equiv a \pmod b]$$

for $n \geq a + b\Delta$. Setting $g(n) = f(n)$, we get

$$f(n) = f\left(\frac{n-a}{b}\right)[n \equiv a \pmod b] + g(n)\,[n < a + b\Delta].$$

for all n.

We obtain

$$\Lambda(s) := \sum_{n>-\alpha} \frac{f(n)}{(n+\alpha)^s} = \frac{G(s)}{1-b^{-s}},$$

where $G(s) = \sum_{-\alpha<n<a+b\Delta} f(n)(n+\alpha)^{-s}$.

Since we do not have to care about Hurwitz ζ functions in this case, the line of integration in the Mellin-Perron formula can be shifted to, e.g., $\Re s = -1$.

Calculation of the asymptotic main term involves calculating $G(0)$, which can be performed by the same ideas as in the proof of Theorem 6.

8. Explicit Formula for the Average Length — Proof of Theorem 7

This section is devoted to the proof of Theorem 7.

In order to determine the sum

$$\sum_{n=1}^{N} \ell(n),$$

we have to consider the three sums

$$\sum_{\substack{n=1\\ n\equiv 0 \pmod 3}}^{N} \ell(n) = \sum_{m=1}^{\lfloor N/3\rfloor} \left(2\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3m+1}{4}\right\rfloor + 1\right)$$

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{N} \ell(n) = \sum_{m=1}^{\lfloor (N-1)/3\rfloor} 2\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3m+2}{2}\right\rfloor$$

and

$$\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{N} \ell(n) = \sum_{m=1}^{\lfloor (N-2)/3\rfloor} \lfloor \log_2(m+1)\rfloor .$$

Sums like

$$\sum_{m=1}^{M} \left\lfloor \tfrac{1}{2}\log_2 \tfrac{3m+1}{4}\right\rfloor$$

can be determined by means of partial summation (cf. [7, § 1.2.4, Ex. 42 (b)]):

$$\begin{aligned}\sum_{m=1}^{M} \left\lfloor \tfrac{1}{2}\log_2 \tfrac{3m+1}{4}\right\rfloor &= M\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3M+1}{4}\right\rfloor - \sum_{k=1}^{M-1} k\left(\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3k+4}{4}\right\rfloor - \left\lfloor \tfrac{1}{2}\log_2 \tfrac{3k+1}{4}\right\rfloor\right)\\ &= M\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3M+1}{4}\right\rfloor - \sum_{k=1}^{M-1} k\left[k = \tfrac{4(4^a-1)}{3} \text{ for some } a\in\mathbb{N}\right]\\ &= M\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3M+1}{4}\right\rfloor - \sum_{a=1}^{\left\lfloor \frac{1}{2}\log_2 \frac{3M+1}{4}\right\rfloor} \tfrac{4(4^a-1)}{3}\\ &= \left(M+\tfrac{4}{3}\right)\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3M+1}{4}\right\rfloor - \tfrac{16}{9}\left(4^{\left\lfloor \frac{1}{2}\log_2 \frac{3M+1}{4}\right\rfloor} - 1\right).\end{aligned}$$

Noting that

$$\left\lfloor \tfrac{1}{2}\log_2 \tfrac{3\lfloor N/3\rfloor+1}{4}\right\rfloor = \left\lfloor \tfrac{1}{2}\log_2 \tfrac{N+1}{4}\right\rfloor$$

for all positive integers N, we obtain the following expression for the first sum:

$$\sum_{\substack{n=1\\ n\equiv 0 \pmod 3}}^{N} \ell(n) = 2\left(\left\lfloor \tfrac{N}{3}\right\rfloor + \tfrac{4}{3}\right)\left\lfloor \tfrac{1}{2}\log_2 \tfrac{N+1}{4}\right\rfloor - \tfrac{32}{9}\left(4^{\left\lfloor \frac{1}{2}\log_2 \frac{N+1}{4}\right\rfloor} - 1\right) + \left\lfloor \tfrac{N}{3}\right\rfloor .$$

The two other sums are determined in an analogous manner, yielding the explicit formula

$$\sum_{n=1}^{N}\ell(n)=2\left(\left\lfloor\tfrac{N}{3}\right\rfloor+\tfrac{4}{3}\right)\left\lfloor\tfrac{1}{2}\log_2\tfrac{N+1}{4}\right\rfloor+2\left(\left\lfloor\tfrac{N-1}{3}\right\rfloor+\tfrac{5}{3}\right)\left\lfloor\tfrac{1}{2}\log_2\tfrac{N+1}{2}\right\rfloor+\left\lfloor\tfrac{N+4}{3}\right\rfloor\left\lfloor\log_2\tfrac{N+1}{3}\right\rfloor+\left\lfloor\tfrac{N}{3}\right\rfloor$$
$$-\tfrac{32}{9}\cdot 4^{\left\lfloor\frac{1}{2}\log_2\frac{N+1}{4}\right\rfloor}-\tfrac{16}{9}\cdot 4^{\left\lfloor\frac{1}{2}\log_2\frac{N+1}{2}\right\rfloor}-2\cdot 2^{\left\lfloor\log_2\frac{N+1}{3}\right\rfloor}+\tfrac{22}{3},$$

which can be rewritten to (2.17).

9. Conclusion

There are several interesting questions that were not addressed in this paper, in order to keep the length of the paper within a reasonable range. We mention: counting occurrences of blocks (counting digits is the special case of blocks of length 1), negative digits, and larger digit sets, leading to redundant representations. We leave these for further research.

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