GENERATING NORMAL NUMBERS OVER GAUSSIAN INTEGERS

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Abstract. In this paper we consider normal numbers over Gaussian integers generated by polynomials. This corresponds to results by Nakai and Shiokawa in the case of real numbers. A generalization of normal numbers to Gaussian integers and their canonical number systems, which were characterized by Kátai and Szabó, is given. With help of this we construct normal numbers of the form

$$0.\lfloor f(z_1)\rfloor\,\lfloor f(z_2)\rfloor\,\lfloor f(z_3)\rfloor\,\lfloor f(z_4)\rfloor\,\lfloor f(z_5)\rfloor\,\lfloor f(z_6)\rfloor\ldots$$

Here we denote by $\lfloor x\rfloor$ the expansion of the integer part of x with respect to a given number system in $\mathbb{Z}[i]$, by f we denote a polynomial with complex coefficients, and by z_i we denote a numbering of the Gaussian integers. We are able to show, that a number, which is constructed in this way, is normal to the given number system.

1. Introduction

When considering number systems one is interested, especially in uniqueness, periodicity and randomness properties of representations. In this paper we deal with the last property, in particular, we are concerned with the distribution of blocks in an expansion. We will call a number *normal* in a number system if every possible block of finite size occurs asymptotically with the same frequency.

For number systems over the reals this has been studied for a very long time. The quantitative aspect is that almost every real number is normal with respect to the Lebesgue measure. But we still do not know whether π or $\log 2$ is normal in a given base $q\geq 2$.

On the other hand we know how to construct normal numbers. This started with the construction of Champernowne who was able to show that

$$0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\ldots$$

is normal in the decimals. This idea was successively extended to the integer part of polynomials over the positive integers by Davenport and Erdös [3] (polynomials with integer coefficients), Schiffer [24] (polynomials with rational coefficients), and Nakai and Shiokawa [22] (polynomials with real coefficients). Finally it was shown by Madritsch *et al.* [20] that

$$0.\lfloor f(1)\rfloor\,\lfloor f(2)\rfloor\,\lfloor f(3)\rfloor\,\lfloor f(4)\rfloor\,\lfloor f(5)\rfloor\,\lfloor f(6)\rfloor\ldots$$

is normal if f is an entire function of bounded logarithmic order and $\lfloor x\rfloor$ denotes the expansion of the integer part of x with respect to a given base $q\geq 2$.

In this paper our aim is to generalize the above mentioned construction of normal numbers to number systems for Gaussian integers. The properties of these number systems have been investigated for instance by Kátai and Szabó [14] as well as Grabner *et al.* [7].

2. Definitions of number systems and normality

We start by defining a number system which will give us the background throughout the whole paper. These definitions are well-known in this area and we recall them mainly following [6].

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Let $b \in \mathbb{Z}[i]$ and let $\mathcal{D}$ be a complete set of residue classes modulo b. Then we call $(b, \mathcal{D})$ a *number system* (NS), if every $z \in \mathbb{Z}[i]$ has a *unique* and *finite* representation

$$z = \sum_{k=0}^{\ell-1} d_k(z) b^k, \tag{2.1}$$

where $d_k(z) = 0$ for $k \geq \ell$. We call $d_k(z) \in \mathcal{D}$ the *digits* of z.

Furthermore b is called the *base* and $\mathcal{D}$ the *set of digits* of the NS. By $\ell(z) := \max\{k : d_{k-1}(z) \neq 0\}$ we denote the length of the expansion. If $\mathcal{D} = \{0, 1, \ldots, |N(b)| - 1\}$ where $N(b)$ denotes the norm of b over $\mathbb{Q}$ then we call $(b, \mathcal{D})$ a *canonical number system* (CNS).

One of the first who considered the possible bases for a CNS was Knuth [15], who was able to show, that $b = -1 \pm i$ is a base. Later this was generalized by Kátai and Szabó [14] who proved that $b = -n \pm i$ with $n \in \mathbb{N}$ is the set of all possible bases for the Gaussian integers. This was further generalized to algebraic number fields and matrix number systems in a series of papers, *cf.* for instance [1, 8, 12, 13, 14, 17, 18, 21, 23].

In order to define uniform distribution and normal numbers we need an equivalent for the "reals". Therefore we will extend our number system onto $\mathbb{C}$. By Theorem 2 in [14] we get that every $\gamma \in \mathbb{C}$ has a (not necessarily unique) representation of the shape

$$\gamma = \sum_{k=-\infty}^{\ell(\gamma)} d_k(\gamma) b^k \quad (d_k(\gamma) \in \mathcal{D}).$$

Then we denote by

$$\lfloor \gamma \rfloor_b = \lfloor \gamma \rfloor := \sum_{k=0}^{\ell(\gamma)} d_k(\gamma) b^k,$$

the *integer* with respect to base b.

We fix a base b and are left with the definition of normal numbers. Therefore let $d_1 \ldots d_l \in \mathcal{D}^l$ be a block of digits of length l. We denote by $\mathcal{N}(\theta; d_1 \ldots d_l; N)$ the number of occurrences of the block $d_1 \ldots d_l$ in the first N digits of θ. Thus

$$\mathcal{N}(\theta; d_1 \ldots d_l; N) := \#\{1 \leq n \leq N : d_1 = d_n(\theta), \ldots, d_l = d_{n+l-1}(\theta)\}.$$

Now we call θ *normal in* $(b, \mathcal{D})$ if for every $l \geq 1$ we have that

$$\mathcal{R}_N(\theta) = \mathcal{R}_{N,l}(\theta) := \sup_{d_1 \ldots d_l} \left| \frac{1}{N} \mathcal{N}(\theta; d_1 \ldots d_l; N) - \frac{1}{|\mathcal{D}|^l} \right| = o(1) \tag{2.2}$$

where the supremum is taken over all possible blocks $d_1 \ldots d_l \in \mathcal{D}^l$ of length l.

By our competion of $\mathbb{Q}(i)$ to $\mathbb{C}$ we get that there can be more than one representation for a $\gamma \in \mathbb{C}$ (*cf.* [10, 11, 21]). We call a $\gamma \in \mathbb{C}$ *ambiguous* if

$$\gamma = \sum_{k=-\infty}^{\ell(\gamma)} x_k b^k = \sum_{k=-\infty}^{\ell(\gamma)} y_k b^k$$

with $x_k \neq y_k$ for at least one $k \leq \ell(\gamma)$. We deal with these numbers in the following lemma.

Lemma 2.1 ([19, Proposition])**.** *Let $(b, \mathcal{D})$ be a CNS. Then every number with an ambiguous representation is not normal.*

As we want to construct a normal number as a concatenation of digital expansions of a certain sequence of numbers we have to give an ordering for the Gaussian integers which will fit our purpose. Therefore we set $q := N(b)$ where N denotes the Norm of b over $\mathbb{Q}$ and let τ be a bijection between $\mathcal{D}$ and $\{0, 1, \ldots, q-1\}$ with $\tau(0) = 0$. Then we extend τ to the Gaussian integers by setting $\tau(d_0 + d_1 b + d_2 b^2 + \cdots + d_k b^k) := \tau(d_0) + \tau(d_1) q + \tau(d_2) q^2 + \cdots + \tau(d_k) q^k$. Furthermore we pull back the relation $\leq$ from $\mathbb{N}$ to $\mathbb{Z}[i]$ by

$$a \leq b :\Leftrightarrow \tau(a) \leq \tau(b), \quad a, b \in \mathbb{Z}[i].$$

By this we define a sequence $\{z_n\}_{n \geq 1}$ of elements of $\mathbb{Z}[i]$ such that $z_n := \tau^{-1}(n-1)$.

For a function $f:\mathbb{Z}[i]\to\mathbb{C}$ we define

$$\theta_b(f) := \theta(f) = \lfloor f(z_1)\rfloor\, q^{-\ell(f(z_1))} + \lfloor f(z_2)\rfloor\, q^{-\ell(f(z_1))-\ell(f(z_2))} + \cdots .$$

This is simply the concatenation of the integer parts of the function values evaluated on the sequence $\{z_n\}_{n\geq 1}$ of Gaussian integers. We are now in a position to state our main theorem.

Theorem 2.2. *Let $f(z)=\alpha_d z^d+\cdots+\alpha_1 z+\alpha_0$ be a polynomial with coefficients in $\mathbb{C}$. Let $(b,\mathcal{D})$ be a CNS in the Gaussian integers. Then for every $l\geq 1$*

$$\sup_{d_1\ldots d_l}\left|\frac{1}{N}\mathcal{N}(\theta_b(f);d_1\ldots d_l;N)-\frac{1}{|\mathcal{D}|^l}\right| = (\log N)^{-1},$$

where the supremum is taken over all blocks of length l.

3. Preliminary Lemmata

The first lemma will help us to rewrite the asymptotics.

Lemma 3.1 ([20, Lemma 3.4])**.** *Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences with $0<a_n\leq b_n$ for all n and*

$$\lim_{n\to\infty}\frac{a_n}{b_n}=0.$$

Then

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}=0.$$

As we deal with blocks of a certain length we need information about the connection of the norm of a Gaussian integer and the length of its expansion. This connection is described by the following lemma.

Lemma 3.2 ([7, Proposition 2.6])**.** *Let $(b,\mathcal{D})$ be a number system in the Gaussian integers and $q:=N(b)$. Then the estimate*

$$\left|\ell(z)-\log_q|z|^2\right|\leq c_b,$$

where $\log_q$ is the logarithm in base q, holds for a certain constant c_b depending only on the base b.

In the proof of our main result we will need the discrepancy (see [5, p.5] for a definition) $D_N(\boldsymbol{x}_n)$ of the first N elements of a sequence $\{\boldsymbol{x}_n\}_{n\geq 1}$ of elements in $\mathbb{R}^2$. The following result will provide us with an estimation of the discrepancy.

Lemma 3.3 (Erdös-Turan-Koksma inequality, [5, Theorem 1.21])**.** *Let $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N$ be points in $\mathbb{R}^2$ and T an arbitrary positive integer. Then*

$$D_N(\boldsymbol{x}_n)\leq\left(\frac{3}{2}\right)^k\left(\frac{2}{V+1}+\sum_{0<\|\boldsymbol{v}\|_\infty\leq V}\frac{1}{r(\boldsymbol{v})}\left|\frac{1}{N}\sum_{n=1}^N e(\boldsymbol{v}\cdot\boldsymbol{x}_n)\right|\right),$$

where $r(\boldsymbol{v})=(\max\{1,|v_1|\})\cdot(\max\{1,|v_2|\})$ for $\boldsymbol{v}=(v_1,v_2)\in\mathbb{Z}^2$.

For the transformation of an exponential sum into an integral we will apply the two following lemmata.

Lemma 3.4 ([2, Lemma 5.4])**.** *Suppose that $F(x_1,\ldots,x_r)$ is a real differentiable function for $0\leq x_j\leq P_j$, $P_j\leq P$ $(j=1,\ldots,r)$, inside the interval of variation of the variables, the function $\partial F(x_1,\ldots,x_r)/\partial x_j$ is piecewise monotone and of constant sign in each of the variables x_j $(j=1,\ldots,r)$ for any fixed values of the other variables, and the number of intervals of monotonicity and constant sign does not exceed s. Next, let the inequalities*

$$\left|\frac{\partial F(x_1,\ldots,x_r)}{\partial x_j}\right|\leq\delta,\quad j=1,\ldots,r,$$

hold for $0<\delta<1$*. Then*

$$\sum_{x_1=0}^{P_1}\cdots\sum_{x_r=0}^{P_r} e(F(x_1,\dots,x_r))$$
$$=\int_0^{P_1}\cdots\int_0^{P_r} e(F(x_1,\dots,x_r))\mathrm{d}x_1\dots\mathrm{d}x_r+\theta_1 rsP^{r-1}\left(3+\frac{2\delta}{1-\delta}\right),$$

where $|\theta_1|\leq 1$*.*

Lemma 3.5 ([25, Lemma 4.2])**.** *Let* $F(x)$ *be a real differentiable function such that* $F'(x)$ *is monotonic, and* $F'(x)\geq m>0$*, or* $F'(x)\leq -m<0$*, throughout the interval* $[a,b]$*. Then*

$$\left|\int_a^b e(F(x))\mathrm{d}x\right|\leq\frac{4}{m}.$$

In the next lemma we give an application of the preceding ones.

Lemma 3.6. *Let* M *and* N *be positive integers with* $M\ll N$*. Let* $F:\mathbb{C}\to\mathbb{C}$ *be a function such that the conditions of Lemma 3.4 and Lemma 3.5 are fulfilled. Then*

$$\sum_{M\leq|z|^2<M+N} e(\operatorname{tr}F(z))\ll\frac{\sqrt{N}}{m}+\frac{N}{(\log N)^{\sigma/2}}+s\left(\frac{3-\delta}{1-\delta}\right)\sqrt{N(\log N)^\sigma}$$

holds for any positive real number σ*. Here* $\operatorname{tr}(x)$ *denotes the trace of an element* $x\in\mathbb{Z}[i]$*.*

Proof. This is a generalization of [6, Lemma 2.1 and 2.2]. In order to apply the two lemmas above we start considering squares in the annulus $M\leq|z|^2<M+N$. Therefore we denote by $D_\nu:=\{z=x+iy\in\mathbb{Z}[i]:-\nu\leq x,y\leq\nu\}$. Now we get by an application of Lemma 3.4 that

$$\begin{aligned}\sum_{z\in D_\nu} e(\operatorname{tr}F(z))&=\sum_{x=-\nu}^{\nu}\sum_{y=-\nu}^{\nu} e(\operatorname{tr}F(x+iy))\\&=\int_{-\nu}^{\nu}\int_{-\nu}^{\nu} e(\operatorname{tr}F(x+iy))\mathrm{d}x\mathrm{d}y+2\theta_1 s\nu\left(\frac{3-\delta}{1-\delta}\right)\end{aligned}$$

We take the modulus in order to apply Lemma 3.5. Thus

$$\begin{aligned}\left|\sum_{z\in D_\nu} e(\operatorname{tr}F(z))\right|&\leq\int_{-\nu}^{\nu}\left|\int_{-\nu}^{\nu} e(\operatorname{tr}F(x+iy))\mathrm{d}x\right|\mathrm{d}y+2\theta_1 s\nu\left(\frac{3-\delta}{1-\delta}\right)\\&\leq 2\nu\max_{-\nu\leq y\leq\nu}\left|\int_{-\nu}^{\nu} e(\operatorname{tr}F(x+iy))\mathrm{d}x\right|+2\theta_1 s\nu\left(\frac{3-\delta}{1-\delta}\right)\\&\leq\frac{8\nu}{m}+2\theta_1 s\nu\left(\frac{3-\delta}{1-\delta}\right)\end{aligned}$$

Secondly we tessellate the annulus $M\leq|z|^2<M+N$ by squares of side length $\sqrt{N/(\log N)^\sigma}$. We define two sets I and B containing the squares which are completely inside the annulus and those which intersect the boundary, respectively. Then we denote by C_I and C_B their contribution to the sum, respectively. There are $\mathcal{O}((\log N)^\sigma)$ squares in I and together with our considerations above we get that

$$C_I\ll\frac{N}{m}+s\left(\frac{3-\delta}{1-\delta}\right)\sqrt{N(\log N)^\sigma}.$$

For the boundary we get that there are two annuli of width $\mathcal{O}(\sqrt{M/(\log M)^\sigma})$ and $\mathcal{O}(\sqrt{(M+N)/(\log M+N)^\sigma})$ that cover the boundary. By noting that $M\ll N$ we get that

$$C_B\ll\frac{N}{(\log N)^{\sigma/2}}.$$

This together with the estimation above yields the result. □

Finally we need an estimation for a complete exponential sum in the Gaussian rationals.

Lemma 3.7 ([9, Theorem 1]). *Let f be a k-th degree polynomial with coefficients in $\mathbb{Q}(i)$ and q be the least common multiple of its coefficients. If $\Lambda(q)$ is a complete set of residues modulo q, then, for any $\varepsilon > 0$,*

$$\sum_{\lambda\in\Lambda(q)} e(\operatorname{tr}(f(\lambda))) \ll (N(q))^{1-\frac{1}{k}+\varepsilon}$$

holds, where the implied constant depends only on f and ε.

4. Properties of the Fundamental Domain

In this section we mainly follow the paper of Gittenberger and Thuswaldner [6]. Let $b = -n+i$ be a base of a CNS in $\mathbb{Z}[i]$. Then every $\gamma \in \mathbb{C}$ has a unique representation of the shape $\gamma = \alpha + \beta b$ with $\alpha, \beta \in \mathbb{R}$. Thus we define the mapping

$$\varphi : \mathbb{C} \to \mathbb{R}^2, \quad \alpha + \beta b \mapsto (\alpha, \beta).$$

As $(1, b)$ is an integral basis we get that $\varphi(\mathbb{Z}[i]) = \mathbb{Z}^2$.

We define the *fundamental domain* $\mathcal{F}'$ to consist of all numbers with zero in the integer part of their b-ary representation. Thus

$$\mathcal{F}' := \left\{ \gamma \in \mathbb{C} \middle| \gamma = \sum_{k\geq 1} d_k b^{-k}, d_k \in \mathcal{D} \right\}.$$

As it is more easy to consider the properties in $\mathbb{R}^2$ we use our embedding from above to switch from $\mathbb{C}$ to $\mathbb{R}^2$. Then we get

$$\mathcal{F} := \varphi(\mathcal{F}') = \left\{ \gamma \in \mathbb{R}^2 \middle| \gamma = \sum_{k\geq 1} d_k B^{-k}, d_k \in \varphi(\mathcal{D}) \right\}$$

where B is the matrix corresponding to the multiplication by b in $\mathbb{R}^2$ given by

$$B = \begin{pmatrix} 0 & -1-n^2 \\ 1 & -2n \end{pmatrix}.$$

(We refer the reader to [23] for more details).

Now we define for every $a \in \mathbb{Z}[i]$ the domain corresponding to the elements of $\mathcal{F}$ whose digit representation after the comma starts with the digits of the expansion of a. In particular, we set

$$\mathcal{F}_a = B^{-\ell(a)}(\mathcal{F} + \varphi(a)). \tag{4.1}$$

As in the case of normal numbers in the reals we need an Urysohn-function for this fundamental domain of numbers starting with a. In the reals we use a lemma due to Vinogradov (cf. Lemma 2 of [26, p.196]), in $\mathbb{C}$, however, we have to construct a corresponding version of this lemma.

For $a \in \mathcal{D}$ this has been done by Gittenberger and Thuswaldner in section 3 of [6]. As the generalization of their construction to the case of $a \in \mathbb{Z}[i]$ runs along the same lines we only state the corresponding results and leave their proofs to the reader.

Lemma 4.1 ([6, Lemma 3.1]). *For all $a \in \mathbb{Z}[i]$ and all $k \in \mathbb{N}$ there exists an axe-parallel tube $P_{k,a}$ with the following properties:*

1. $\partial\mathcal{F}_a \subset P_{k,a}$ *for all* $k \in \mathbb{N}$,
2. $\lambda_2(P_{k,a}) = \mathcal{O}(\mu^k / |b|^{2k})$,
3. $P_{k,a}$ *consists of* $\mathcal{O}(\mu^k)$ *axe-parallel rectangles with* $1 < \mu < |b|^2$, *each of which has Lebesgue measure* $\mathcal{O}(|b|^{-2k})$.

Here we denote by λ_2 the usual Lebesgue measure of $\mathbb{R}^2$.

In the proof of Gittenberger and Thuswaldner [6] they define for every pair (k,a) suitable axe-parallel polygons $\Pi_{k,a}$. Then they get that $d(\Pi_{k,a},\partial\mathcal{F}_a) < c\,|b|^{-k}$, for a constant $c>0$, where $d(\cdot,\cdot)$ denotes the Hausdorff metric, and

$$P_{k,a} := \left\{ z\in\mathbb{R}^2 \,\middle|\, \|z-\Pi_{k,a}\|_\infty \leq 2c\,|b|^{-k} \right\}. \tag{4.2}$$

As in [6] we denote by $I_{k,a}$ the interior of $\Pi_{k,a}$ and define f_a by

$$f_a(x,y) = \frac{1}{\Delta^2}\int_{-\Delta/2}^{\Delta/2}\int_{-\Delta/2}^{\Delta/2}\psi_a(x+\bar{x},y+\bar{y})d\bar{x}d\bar{y}, \tag{4.3}$$

where

$$\Delta := 2c_\Delta\,|b|^{-k} \tag{4.4}$$

with $c_\Delta>0$ a constant and

$$\psi_a(x,y) = \begin{cases} 1 & \text{if } (x,y)\in I_{k,a}\\ \frac12 & \text{if } (x,y)\in \Pi_{k,a}\\ 0 & \text{otherwise.}\end{cases}$$

Now f_a is the desired Urysohn function for $\mathcal{F}_a$ in $\mathbb{R}^2$. We perform Fourier analysis of this function and get the following results for its coefficients.

Lemma 4.2 ([6, Lemma 3.2]). *Let $f_a(x,y)=\sum_{m,n}C(m,n)e(mx+ny)$ be the Fourier expansion of f_a. Then for the Fourier coefficients $C(m,n)$ we get the estimates*

$$C(m,n) = \begin{cases} |b|^{-2\ell(a)} & m=n=0,\\ \mu^k c(m)c(n) & \textit{otherwise,}\end{cases} \tag{4.5}$$

where

$$c(t) \ll \begin{cases} 1 & t=0,\\ \min(|t|^{-1},\Delta\,|t|^{-2}) & \textit{otherwise.}\end{cases} \tag{4.6}$$

As the proof of this lemma runs along the same lines as that of [6, Lemma 3.2] we omit it.

The coefficient $C(0,0)$ will correspond to the main term and all others contribute to the error term. One of our main tools will be Weyl sums which will be discussed in the next section.

5. The Weyl Sum

This estimation will play a crucial rôle in the proof of the Theorem.

Throughout this section we denote by f a polynomial with coefficients in $\mathbb{C}$. Thus

$$f(z) = \alpha_d z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z.$$

In order to establish an upper bound we will generalize Lemma 2 of Nakai and Shiokawa [22].

Proposition 5.1. *Let $G>0$ be any constant and $N\geq 2$. Let s be an integer with $1\leq s\leq d$, let H_i,K_i $(i=s+1,s+2,\dots,d-1,d)$ be any positive constants, and let H_s^*, K_s^* be constants such that*

$$H_s^* \geq 2^{3(s+2)} + 2^{s+3}(G+\max_{s<i\leq d}H_i) + s\sum_{i=s+1}^d K_i,$$

$$K_s^* \geq 2^{3(s+2)} + 2^{s+3}(G+\max_{s<i\leq d}H_i) + 2s\sum_{i=s+1}^d K_i.$$

Suppose that there are Gaussian integers a_i and q_i $(s<i\leq d)$ such that

$$1\leq |q_i|^2 \leq (\log N)^{K_i} \quad \textit{and} \quad \left|\alpha_i - \frac{a_i}{q_i}\right| \leq \frac{(\log N)^{H_i}}{q_i N^{i/2}}$$

and that there exist no Gaussian integers a_s *and* q_s *with* $(a_s, q_s) = 1$ *such that*

$$1 \le |q_s|^2 \le (\log N)^{K_s^*} \quad \text{and} \quad \left|\alpha_s - \frac{a_s}{q_s}\right| \le \frac{(\log N)^{H_s^*}}{q_s N^{s/2}}. \tag{5.1}$$

Then

$$\left|\sum_{|z|^2 \le N} e(\operatorname{tr}(f(z)))\right| \ll N(\log N)^{-G}$$

holds.

Before we start proving the proposition we need two lemmata. The first deals with approximation by Gaussian integers.

Lemma 5.2 ([4, Theorem 4.5])**.** *Given any* $z = x + iy \in \mathbb{C}$ *and* $N \in \mathbb{N}$*, there exist Gaussian integers* a *and* q *with* $0 < |q|^2 \le N$ *such that*

$$\left|z - \frac{a}{q}\right| < \frac{2}{|q|\sqrt{N}}.$$

Furthermore we need a lemma which considers the case that $s = d$, the degree of the polynomial f, *i.e.*, that the leading coefficient is already well approximable.

Lemma 5.3 ([6, Proposition 2.1])**.** *Let* $(h, q) = 1$ *and*

$$g(x) = \frac{h}{q}x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x + \alpha_0$$

where $(\log N)^H \le |q|^2 \le N^d(\log N)^{-H}$*. Then we have*

$$\left|\sum_{|z|^2 < N} e\left(\operatorname{tr}(g(z))\right)\right| \ll N(\log N)^{-G}$$

with $H \ge 2^{d+2}G + 2^{3(d+2)}$*.*

In order recursively apply the Lemma 5.3 we need a tool to rewrite it.

Lemma 5.4 ([16, Lemma 26])**.** *Let functions* $f_1(x)$ *and* $f_2(x)$ *be defined for* $x \in M$*. Then*

$$\sum_{x \in M} e(f_1(x) + f_2(x)) = \sum_{x \in M} e(f_1(x)) + 2\pi\theta \sum_{x \in M} |f_2(x)|,$$

where $|\theta| \le 1$*.*

Corollary 5.5. *Let* $g(x) = \alpha_d x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{C}[X]$*. If there exist* $h, q \in \mathbb{Z}[X]$ *such that* $(h, q) = 1$ *and*

$$\left|\alpha_d - \frac{h}{q}\right| \le \frac{(\log N)^H}{|q|\, N^{\frac{d}{2}}}$$

with $(\log N)^H \le |q| \le N^{\frac{d}{2}}(\log N)^{-H}$ *and* $H \ge 2^{d+1}G + 2^{3(d+2)-1}$*, then we have*

$$\left|\sum_{|z|^2 < N} e\left(\operatorname{tr}(g(z))\right)\right| \ll N(\log N)^{-G}.$$

Proof. This easily follows from

$$\left|\alpha_d - \frac{h}{q}\right| \le \frac{(\log N)^H}{|q|\, N^{\frac{d}{2}}} \le N^{-\frac{d}{2}}$$

together with an application of Lemma 5.4. Thus we get by Lemma 5.3 that

$$\begin{aligned}
&\left|\sum_{|z|^2<N} e\left(\operatorname{tr}(g(z))\right)\right| \\
&\leq \left|\sum_{|z|^2<N} e\left(\operatorname{tr}\left(\frac{h}{q}z^d+\alpha_{d-1}z^{d-1}+\cdots+\alpha_0\right)\right)\right| + \sum_{|z|^2<N}\left|\operatorname{tr}\left(\left(\alpha_d-\frac{h}{q}\right)z^d\right)\right| \\
&\ll N(\log N)^{-G}+N^{\frac12}.
\end{aligned}$$

□

Now we can start the proof of Proposition 5.1.

Proof of Proposition 5.1. This proof mainly follows the ideas of Nakai and Shiokawa for their proof of Lemma 2 in [22].

We consider the different possibilities for s. If $s=d$ nothing is to show as this is exactly the case of Corollary 5.5.

Let $s<d$. We denote by k the least common multiple of $q_{s+1},\dots,q_d$. We have $k\in\mathbb{Z}[i]$ because the Gaussian integers are a unique factorization domain. We denote by Q the integer such that $|k|^2 Q\leq N<|k|^2(Q+1)$. By our assumptions we have that

$$1\leq |k|^2\leq (\log N)^K \quad \text{with } K=\sum_{i=s+1}^{d} K_i$$

and

$$N(\log N)^{-K}\ll Q\ll N/|k|^2.$$

Now we use the fact that $\mathbb{Z}[i]$ is an Euclidean domain. From this we get that for every $s\in\mathbb{Z}[i]$ there exist unique $q,r\in\mathbb{Z}[i]$ with $|r|^2<|k|^2$ such that $s=qk+r$. Thus we get that there exists a complete residue system R modulo k with

$$R\subset\{z\in\mathbb{Z}[i]:|z|\leq|k|\}.$$

We use this residue system to tessellate the open disc $D:=\{z:|z|^2<N\}$ with translates of R. Let T be these translates, *i.e.*,

$$T:=\{t\in\mathbb{Z}[i]:(R+tk)\cap D\neq\emptyset\}.$$

Now we define I to be the translates which are completely contained in D, *i.e.*,

$$I:=\{t\in T:(R+tk)\subset D\}.$$

As there are $\mathcal{O}(\sqrt{N})$ points on the circumference and there are $\mathcal{O}(|k|)$ points in R we get that

$$\sum_{|z|^2\leq N} e(\operatorname{tr}(f(z)))=\sum_{t\in I}\sum_{r\in R} e(\operatorname{tr}(f(tk+r)))+\mathcal{O}(\sqrt{N}\,|k|).$$

As in the proof of Lemma 2 of Nakai and Shiokawa in [22] we want to do Abel Summation. Therefore we need an ordering on I. Let $x,y\in I$, then define

$$x\prec y :\Leftrightarrow \begin{cases} |x|<|y| \text{ or}\\ (|x|=|y| \text{ and } \arg(x)<\arg(y))\end{cases}$$

By the polar representation of every complex number we get that this ordering is well defined. Furthermore we set $\sigma:\mathbb{N}\to I$ a bijection such that $\sigma(1)=0$, $\sigma(|I|)=\max I$, and

$$\sigma(x)\prec\sigma(y):\Leftrightarrow x<y,$$

where the maximum is with respect to $\prec$. Let $M=|I|$ then we have

$$\sum_{|z|^2\leq N} e(\operatorname{tr}(f(z)))=\sum_{n=1}^{M}\sum_{r\in R} e(\operatorname{tr}(f(\sigma(n)k+r)))+\mathcal{O}(\sqrt{N}\,|k|) \tag{5.2}$$

Now we are ready to do Abel Summation and define for short

$$\psi_r(x) = \sum_{i=s+1}^{d} \gamma_i (xk+r)^i, \qquad \gamma_i = \alpha_i - \frac{a_i}{q_i},$$

$$\varphi_r(x) = \sum_{i=1}^{s} \alpha_i (xk+r)^i, \qquad T_r(\ell) = \sum_{n=1}^{\ell} e(\mathrm{tr}(\varphi_r(\sigma(n)))).$$

By the linearity of the trace tr we get that

$$\begin{aligned}
&\sum_{n=1}^{M}\sum_{r\in R} e(\mathrm{tr}(f(\sigma(n)k+r)))\\
&= \sum_{r\in R}\sum_{n=1}^{M} e(\mathrm{tr}(\sum_{i=1}^{d}\alpha_i(\sigma(n)k+r)^i))\\
&= \sum_{r\in R}\sum_{n=1}^{M} e(\mathrm{tr}(\sum_{i=1}^{s}\alpha_i(\sigma(n)k+r)^i + \sum_{i=s+1}^{d}\alpha_i(\sigma(n)k+r)^i))\\
&= \sum_{r\in R}\sum_{n=1}^{M} e(\mathrm{tr}(\varphi_r(\sigma(n)) + \sum_{i=s+1}^{d}(\gamma_i+\frac{a_i}{q_i})(\sigma(n)k+r)^i))\\
&= \sum_{r\in R} e\left(\mathrm{tr}\left(\sum_{i=s+1}^{d}\frac{a_i}{q_i}r^i\right)\right)\sum_{n=1}^{M} e(\mathrm{tr}(\varphi_r(\sigma(n))+\psi_r(\sigma(n)))) \qquad (5.3)\\
&= \sum_{r\in R} e\left(\mathrm{tr}\left(\sum_{i=s+1}^{d}\frac{a_i}{q_i}r^i\right)\right)\sum_{n=1}^{M} e(\mathrm{tr}(\psi_r(\sigma(n))))\,(T_r(n)-T_r(n-1))\\
&= \sum_{r\in R} e\left(\mathrm{tr}\left(\sum_{i=s+1}^{d}\frac{a_i}{q_i}r^i\right)\right)\Bigg[e(\mathrm{tr}(\psi_r(\sigma(M+1))))T_r(M)\\
&\quad + \sum_{n=1}^{M}\left(e(\mathrm{tr}(\psi_r(\sigma(n)))) - e(\mathrm{tr}(\psi_r(\sigma(n+1))))\right)T_r(n)\Bigg]\\
&\ll \sum_{r\in R}\left[|T_r(M)| + \sum_{n=1}^{M}\left|e(\mathrm{tr}(\psi_r(\sigma(n)))) - e(\mathrm{tr}(\psi_r(\sigma(n+1))))\right|\,|T_r(n)|\right]
\end{aligned}$$

As the trace is a linear functional we get

$$\frac{d}{dx}\,\mathrm{tr}(f(x)) = \mathrm{tr}\left(\frac{df}{dx}\right).$$

Noting that for $a\in\mathbb{C}$ we have $\mathrm{tr}(a)\ll|a|$ and that for $1<n\le M$ we get $|\sigma(n)-\sigma(n+1)|\ll N^{\frac{1}{2}}$, we apply the mean-value of calculus theorem to get

$$\left|e(\mathrm{tr}(\psi_r(\sigma(n)))) - e(\mathrm{tr}(\psi_r(\sigma(n+1))))\right| \ll |k|\sum_{i=s+1}^{d}|\gamma_i|\,N^{i/2-1} \ll |k|\,\frac{(\log N)^H}{N}$$

where $H=\max\{H_i : i=1,\ldots,s\}$.

Thus

$$\sum_{n=1}^{M}\sum_{r\in R} e(\mathrm{tr}(f(\sigma(n)k+r))) \ll \sum_{r\in R}\left[|T_r(M)| + |k|\,\frac{(\log N)^H}{N}\sum_{n=1}^{M}|T_r(n)|\right]. \tag{5.4}$$

If we can show that

$$|T_r(n)| \ll \frac{N}{|k|\,(\log N)^{G+H}} \tag{5.5}$$

then we are done. We may assume that

$$n \gg \frac{N}{|k| (\log N)^{G+H}}. \tag{5.6}$$

We split the estimation of $T_r(n)$ up, according to whether there exist a and q with $(a,q) = 1$ such that

$$(\log N)^{H'} \leq |q|^2 \leq N^s (\log N)^{-H'} \tag{5.7}$$

and

$$\left| k^s \alpha_s - \frac{a}{q} \right| \leq |q|^2 ,$$

with $H' = 2^{3(s+2)} + 2^{s+3}(G+H) + sK$, or not.

- Suppose there exist such a and q. Then by the definition of H' together with (5.6) we get that
$$(\log n)^{h'} \leq |q|^2 \leq n^s (\log n)^{-h'},$$
where $h' = 2^{3(s+2)} + 2^{s+2}(G+H)$. Thus an application of Lemma 5.3 yields
$$|T_r(n)| \ll n(\log n)^{-(G+H)} \ll \frac{N}{|k| (\log N)^{(G+H)}}.$$
- On the contrary if there are no such a and q then we get by Lemma 5.2 that there must exist a and q with $(a,q) = 1$ and $|q|^2 \leq N^s (\log N)^{-H'}$. Thus we get by (5.7) that
$$1 \leq |q|^2 \leq (\log N)^{-H'} \qquad \text{and} \qquad \left| k^s \alpha_s - \frac{a}{q} \right| \leq \frac{(\log N)^{-H'/2}}{|q| N^{s/2}}.$$
Then, however, we get
$$|k^s q|^2 \leq (\log N)^{H'+sK} \leq (\log N)^{K_s^*},$$
and thus
$$\left| \alpha_s - \frac{a}{k^s q} \right| \leq \frac{(\log N)^{H_s^*}}{|k|^s |q| N^s},$$
which contradicts the assumption on α_s.

Therefore we have shown (5.5). Thus we get together with (5.2) and (5.4) that

$$\begin{aligned}
\sum_{|z|^2 \leq N} e(\operatorname{tr}(f(z))) &\ll \sum_{r \in R} \left[|T_r(M)| + |k| \frac{(\log N)^H}{N} \sum_{n=1}^{M} |T_r(n)| \right] + \sqrt{N} |k| \\
&\ll \sum_{r \in R} \left[\frac{N}{|k| (\log N)^{G+H}} + \frac{1}{(\log N)^G} M \right] + \sqrt{N} |k| \\
&\ll \frac{N}{(\log N)^G}
\end{aligned}$$

and the proposition is proven. □

Now we have enough tools to proceed to the proof of the main theorem.

6. Proof of Theorem 2.2

In the rest of the paper we will consider the proof of Theorem 2.2. The proof will split up into several parts.

(1) We start in Section 6.1 with a definition of several parameters which will be useful in the proof. Furthermore we show some connections between them.
(2) Then in Section 6.2 we rewrite the problem into one of an estimation of an exponential sum. This sum is finally transfered into one of type as in Proposition 5.1 or Lemma 5.3.

(3) We consider these sums according to the b-adic length of their arguments. There will be no problem when considering the middle ones in Section 6.3. By middle we mean that there exists a upper and lower bound for the b-adic length of the expansion. For those arguments with a long or short expansion we have to use different methods in Sections 6.4 and 6.5, respectively.
(4) Finally we put everything together and get the result.

Throughout the proof we will fix N and the block $d_1 \dots d_l$ under consideration. Furthermore we set

$$a := \sum_{i=1}^{l} d_i b^{l-i} \tag{6.1}$$

for abbreviation.

6.1. **Defining parameters and explaining relations between them.** Let m be the unique positive integer such that

$$\sum_{n \leq m-1} \ell(f(z_n)) < N \leq \sum_{n \leq m} \ell(f(z_n)), \tag{6.2}$$

where $z_n := \tau^{-1}(n-1)$ with $n \geq 1$. Furthermore we denote by M the maximum norm and by J the maximum length of the $(b, \mathcal{D})$-ary expansion of $\lfloor f(z_n) \rfloor$ for $1 \leq n \leq m$, *i.e.*,

$$M := \max_{n \leq m} |z_n|^2, \quad J := \max_{n \leq m} \ell(f(z_n)).$$

These will be of central interest for us.

Now we will use Lemma 3.2 to connect m and M. We get

$$\left| \log_{|b|^2} \max_{n \leq m} |z_n|^2 - \ell(\max_{n \leq m} z_n) \right| = \left| \log_{|b|^2} M - \ell(z_m) \right| = \left| \log_{|b|^2} M - \left\lfloor \log_{|b|^2} m \right\rfloor \right| \leq c,$$
$$M \ll\gg m,$$

where $\ll\gg$ means both $\ll$ and $\gg$.

For the connection of M and J we note that $|f(z)| \ll\gg |z|^d$. Thus we get by Lemma 3.2 that

$$\begin{gathered} \left| \log_{|b|^2} \max_{n \leq m} |f(z_n)|^2 - J \right| \ll\gg \left| \log_{|b|^2} \max_{n \leq m} |z_n|^{2d} - J \right| = \left| \log_{|b|^2} M^d - J \right|, \\ M \ll\gg |b|^{2\frac{J}{d}} \leq c. \end{gathered} \tag{6.3}$$

Finally we get the following relation between M and N.

$$N = mJ + \mathcal{O}(m) = c_0 M \log_q M + \mathcal{O}(M),$$

where c_0 is a positive constant depending on d and b.

Next we want to split the sum on the right of (6.2) up into parts where $f(z_n)$ has the same b-ary length. Therefore let $I_l, I_{l+1}, I_{l+2}, \dots, I_J \subset \{1, \dots, m\}$ be such that

$$n \in I_j :\Longleftrightarrow \ell(f(z_n)) \geq j.$$

In order to estimate the size of these subsets we define M_j $(j = l, l+1, \dots, J)$ to be the least integers such that any $z \in \mathbb{C}$ of norm greater or equal M_j has at least length j, *i.e.*,

$$M_j := \max_{\ell(z) < j} |z|^2 = \max_{n < |b|^{2(j-1)}} |z_n|^2.$$

By the same arguments as in (6.3) we get that $M_j \ll\gg |b|^{2\frac{j}{d}}$. Furthermore we set

$$X_j := M - M_j. \tag{6.4}$$

6.2. **Rewriting the problem.** With the help of the parameters defined above we can easily rewrite our problem. Therefore we set $\mathcal{N}(f(z_n))$ the number of occurrences of the block $d_1 \dots d_l$ in the b-ary expansion of the integer part of $\lfloor f(z_n) \rfloor$. Then we get that

$$\left| \mathcal{N}(\theta_q(f); d_1 \dots d_\ell, N) - \sum_{n \leq m} \mathcal{N}(f(z_n)) \right| \leq 2lm.$$

Thus it suffices to show that

$$\sum_{n \leq m} \mathcal{N}(f(z_n)) = \frac{N}{|\mathcal{D}|^l} + \mathcal{O}\left(\frac{N}{\log N}\right). \tag{6.5}$$

In order to count the occurrences of the block $d_1 \dots d_l$ in $\lfloor f(z_n) \rfloor$ properly we define the indicator function of $\mathcal{F}_a$ (where a is as in (6.1) and $\mathcal{F}_a$ is defined in (4.1))

$$\mathcal{I}_a(z) = \begin{cases} 1 & z \in \mathcal{F}_a, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, writing $f(z_n)$ in $(b, \mathcal{D})$-ary expansion for a fixed $n \in \{1, \dots, m\}$, *i.e.*,

$$f(z_n) = a_r b^r + a_{r-1} b^{r-1} + \cdots + a_1 b + a_0 + a_{-1} b^{-1} + \cdots,$$

with $a_i \in \mathcal{D}$ for $i = r, r-1, \dots,$ we see that the function $\mathcal{I}(z_n)$ is defined such that

$$\mathcal{I}(b^{-j-1} f(z_n)) = 1 \iff d_1 \dots d_l = a_{j-1} \dots a_{j-l}.$$

As every I_j $(l \leq j \leq J)$ consists of exactly those $f(z_n)$ whose $(b, \mathcal{D})$-ary expansion has at least length j, we get that

$$\sum_{n \leq m} \mathcal{N}(f(z)) = \sum_{l \leq j \leq J} \sum_{n \in I_j} \mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right).$$

For every j there can be elements $z \in \mathbb{Z}[i]$ with $|z|^2 < M_j$ but $\ell(z) \geq j$. By Lemma 3.2 we get that there are only finitely many with this property. Now by Lemma 3.1 we get that

$$\sum_{n \in I_j} 1 = \sum_{|z_n| < M_j} 1 + \sum_{M_j \leq |z_n|^2 < M} 1 \sim \sum_{M_j \leq |z_n|^2 < M} 1.$$

Therefore we can assume that there are no z with $\ell(z) \geq j$ and $|z|^2 < M_j$.

In order to estimate $\mathcal{I}(z)$ we use our considerations of Section 4. Noting that $\mathcal{F}_a$ can be covered by a set $I_{k,a}$ and an axe parallel tube $P_{k,a}$ (*cf.* (4.2)), we have to consider how often the sequence $\left\{b^{-j-1} f(z_n)\right\}_{n \in I_j}$ hits each of these sets. The first one, $I_{k,a}$, is characterized by the Urysohn function $f_a(x, y)$ (*cf.* (4.3)) and for the axe-parallel tube we define

$$\mathcal{E}_j := \# \left\{ n \in I_j : \varphi\left(\frac{f(z_n)}{b^{j+1}}\right) \in P_{k,a} \right\}.$$

Thus we get for every $j \in \{l, l+1, \dots, J\}$ that

$$\sum_{n \in I_j} \mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right) = \sum_{n \in I_j} f_a\left(\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right) + \mathcal{O}\left(\mathcal{E}_j\right). \tag{6.6}$$

We consider both terms on the right hand side of (6.6) separately starting with f_a and get by Lemma 4.2 that

$$f_a\left(\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right) = |b|^{-2\ell(a)} + \sum_{\mathbf{0} \neq \boldsymbol{v} \in \mathbb{Z}^2} C(v_1, v_2)\, e\left(\boldsymbol{v} \cdot \varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right),$$

where $\boldsymbol{v} = (v_1, v_2)$ and $C(\cdot, \cdot)$ is defined as in (4.5).

By the estimations of the Fourier coefficients in (4.6) we can split the sum up into those $\boldsymbol{v}$ with $\|\boldsymbol{v}\|_\infty \leq \Delta^{-1}$ and the rest. For $\|\boldsymbol{v}\|_\infty > \Delta^{-1}$ we apply our estimate for the coefficients in (4.6) and estimate the $e(\cdot)$ function trivially to get

$$\sum_{n\in I_j} f_a\left(\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right) \ll \frac{X_j}{|b|^{2l}} + X_j\mu^k\Delta^2 + \mu^k \sum_{\mathbf{0}<\|\boldsymbol{v}\|_\infty\leq\Delta^{-1}} \frac{1}{r(\boldsymbol{v})} \sum_{n\in I_j} e\left(\boldsymbol{v}\cdot\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right). \tag{6.7}$$

To estimate $\mathcal{E}_j$ we use the Erdos-Turan-Koksma inequality (Lemma 3.3). By Lemma 4.1 we can split the tube $P_{k,a}$ into a family of μ^k rectangles $\boldsymbol{R}_j$. As the discrepancy is defined on a rectangle (*cf.* [5, p.5]) we get by an application of Lemma 3.3 that

$$\begin{aligned}\mathcal{E}_j &\ll \sum_{\boldsymbol{R}_j} X_j\lambda_2(R) + X_jD_{X_j}(\{x_n\})\\ &\ll X_j\sum_{\boldsymbol{R}_j}\left(\lambda_2(R) + \frac{2}{H+1} + \sum_{0<\|\boldsymbol{h}\|_\infty\leq H}\frac{1}{r(\boldsymbol{v})}\left|\frac{1}{X_j}\sum_{n\in I_j} e\left(\boldsymbol{v}\cdot\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right)\right|\right),\end{aligned} \tag{6.8}$$

where the sum is extended over all rectangles R comprising the tube $P_{k,a}$ can be split into.

By the property (3) of $P_{k,a}$ described in Lemma 4.1 and possible overlappings of the rectangles in $\boldsymbol{R_j}$ we get that

$$\sum_{\boldsymbol{R}_j}\lambda_2(R) \ll \left(\frac{\mu}{|b|^2}\right)^k.$$

Thus (6.8) simplifies to

$$\mathcal{E}_j \ll X_j\left(\left(\frac{\mu}{|b|^2}\right)^k + \frac{\mu^k}{H+1} + \frac{\mu^k}{X_j}\sum_{0<\|\boldsymbol{v}\|_\infty\leq H}\frac{1}{r(\boldsymbol{v})}\sum_{n\in I_j} S(\boldsymbol{v},j)\right). \tag{6.9}$$

As both exponential sums in (6.7) and (6.8) are of the same shape, we define for short

$$S(\boldsymbol{v},j) := \sum_{n\in I_j} e\left(\boldsymbol{v}\cdot\varphi\left(\frac{f(z_n)}{b^{j+1}}\right)\right). \tag{6.10}$$

Plugging (6.7), (6.9), and (6.10) in (6.6) and subtracting the mayor part we get

$$\begin{aligned}&\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right) - \frac{X_j}{|b|^{2l}}\right| \ll\\ &X_j\left(\mu^k\Delta^2 + \frac{2\mu^k}{H+1} + \left(\frac{\mu}{|b|^2}\right)^k\right) + \sum_{\mathbf{0}<\|\boldsymbol{v}\|_\infty\leq\Delta^{-1}}\frac{\mu^k}{r(\boldsymbol{v})}S(\boldsymbol{v},j) + \sum_{0<\|\boldsymbol{v}\|_\infty\leq H}\frac{\mu^k}{r(\boldsymbol{v})}S(\boldsymbol{v},j).\end{aligned} \tag{6.11}$$

In order to transfer the exponential sum from $\mathbb{Z}^2$ to $\mathbb{Z}[i]$ we use the same idea as Gittenberger and Thuswaldner in [6, p.335]. Thus let

$$\tau(z) := (\operatorname{tr} z, \operatorname{tr} bz)^t = \Xi\varphi(z),$$

where $\Xi = VV^t$ an V is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1\\ b & \bar{b}\end{pmatrix}.$$

By this we get that

$$\boldsymbol{v}\cdot\varphi\left(\frac{f(z)}{b^{j+1}}\right) = \boldsymbol{v}\Xi^{-1}\tau\left(\frac{f(z)}{b^{j+1}}\right) = \operatorname{tr}\left((\widetilde{v_1} + b\widetilde{v_2})\frac{f(z)}{b^{j+1}}\right),$$

where $(\widetilde{v_1},\widetilde{v_2}) := \boldsymbol{v}\Xi^{-1}$.

Thus we get that (6.10) transfers to

$$\begin{aligned} S(\boldsymbol{v},j) &= \sum_{n\in I_j} e\left(\operatorname{tr}\left((\widetilde{v_1}+b\widetilde{v_2})\frac{f(z_n)}{b^{j+1}}\right)\right) \\ &\ll \sum_{M_j\leq |z|^2<M_j+X_j} e\left(\operatorname{tr}\left((\widetilde{v_1}+b\widetilde{v_2})\frac{f(z)}{b^{j+1}}\right)\right), \end{aligned} \tag{6.12}$$

where we have used that $|I_j| \ll\gg X_j$ together with the definition of X_j in (6.4).

We assume that we take k and H such that $\Delta^{-1}, H \ll (\log N)$, which is possible since Δ depends on k (*cf.* (4.4)). The value of k and H is chosen later depending on j.

In the following subsections we want to consider the different sums $S(\boldsymbol{v},j)$ according to the size of j. We therefore split the area into three intervals as follows

$$l \leq j \leq l + C_l \log\log N, \tag{6.13}$$

$$l + C_l \log\log N < j \leq J - C_u \log\log N, \tag{6.14}$$

$$J - C_u \log\log N < j \leq J, \tag{6.15}$$

where C_l and C_u are sufficiently large constants.

6.3. **A first estimation of $S(\boldsymbol{v},j)$.** We will start with the j satisfying (6.14).

Assume first that there are two Gaussian integers a and q such that

$$\left|\frac{\widetilde{v_1}+b\widetilde{v_2}}{b^j}\alpha_d - \frac{a}{q}\right| \leq \frac{1}{|q|^2} \quad \text{and} \quad (\log X_j)^H \leq |q|^2 \leq X_j^d (\log X_j)^{-H}, \tag{6.16}$$

with $G = 3$ and $H = 2^{d+2}G + 2^{3(d+2)}$. Then we apply Lemma 5.3 and get

$$S(\boldsymbol{v},j) \ll X_j (\log X_j)^{-G}.$$

Now we will show that (6.16) holds for all j satisfying (6.14).

If (6.16) does not hold, then we get by an application of Lemma 5.2 that there are $a,q\in\mathbb{Z}[i]$ such that

$$(a,q)=1, \quad 1\leq |q|^2 \leq X_j^d(\log X_j)^{-H}, \quad \text{and} \quad \left|\frac{\widetilde{v_1}+b\widetilde{v_2}}{b^j}\alpha_d - \frac{a}{q}\right| \leq \frac{(\log X_j)^H}{|q|\, X_j^{\frac{d}{2}}} \leq \frac{1}{|q|^2}.$$

We distinguish two cases for the size of $|q|^2$. Assume first that $2\leq |q|^2 \leq (\log X_j)^H$. Thus we get

$$\left|\frac{\widetilde{v_1}+b\widetilde{v_2}}{b^j}\alpha_d\right| > \frac{1}{|q|} - \frac{1}{|q|^2} \geq \frac{1}{2\,|q|} \gg (\log X_j)^{-H}$$

and therefore

$$|b|^j \ll |(\widetilde{v_1}+b\widetilde{v_2})\alpha_d|\,(\log X_j)^H \ll (\log N)(\log X_j)^H,$$

which contradicts (6.14) for C_l sufficiently large.

We will denote by $\|z\|$ the distance of the norm of z over $\mathbb{Q}$ to the nearest integer, *i.e.*,

$$\|z\| := \min_{n\in\mathbb{Z}} \left||z|^2 - n\right|.$$

Now if $|q|^2 = 1$ then $q=1$ and $\left\|(\widetilde{v_1}+b\widetilde{v_2})(b^{-j})\alpha_d\right\| < X_j^d(\log X_j)^{-2H}$. If $\left|(\widetilde{v_1}+b\widetilde{v_2})(b^{-j})\alpha_d\right|^2 > \frac{\sqrt{2}}{2}$ then

$$|b|^{2j} \ll |(\widetilde{v_1}+b\widetilde{v_2})\alpha_d| \ll \log N,$$

which contradicts (6.14) for C_l sufficiently large.

On the other hand if $\left|(\widetilde{v_1}+b\widetilde{v_2})b^{-j}\alpha_d\right| < \frac{\sqrt{2}}{2}$ we get that

$$\left|(\widetilde{v_1}+b\widetilde{v_2})b^{-j}\alpha_d\right|^2 = \left\|(\widetilde{v_1}+b\widetilde{v_2})b^{-j}\alpha_d\right\| < X_j^d(\log X_j)^{-2H},$$

which implies that

$$|b|^{2j} \gg |(\widetilde{v_1}+b\widetilde{v_2})\alpha_d|^2\, X_j^d(\log X_j)^{-2H}$$

contradicting our assumption on C_u in (6.14).

Thus for j such that (6.14) holds we get

$$S(\boldsymbol{v},j) \ll X_j(\log X_j)^{-G}. \tag{6.17}$$

Plugging this into (6.11) we get that

$$\begin{aligned}\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{|b|^{2l}}\right| \ll & X_j\left(\mu^k\Delta^2+\frac{\mu^k}{V+1}+\left(\frac{\mu}{|b|^2}\right)^k\right.\\ & \left.+\frac{\mu^k}{(\log X_j)^3}\left\{\sum_{\mathbf{0}<\|\boldsymbol{v}\|_\infty\leq\Delta^{-1}}+\sum_{0<\|\boldsymbol{v}\|_\infty\leq V}\right\}\frac{1}{r(\boldsymbol{v})}\right)\end{aligned} \tag{6.18}$$

Now we can choose k and H under the assumption that both are $\ll (\log N)$. Thus we set for j as in (6.14) together with the definition of Δ in (4.4) that

$$k := C_k \log\log X_j, \quad H := \mu^k \log X_j, \quad \Delta^{-1} = \frac{(\log X_j)^{C_k \log|b|}}{2c_\Delta}, \tag{6.19}$$

for C_k an arbitrary constant. Furthermore we define $C_\mu > 1$ to be such that

$$C_\mu \mu = |b|^2 .$$

By our setting we get for j as in (6.14) that

$$\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{|b|^{2l}}\right| \ll X_j\left((\log X_j)^{-1}+(\log X_j)^{-2}(\log\log X_j)^2\right) \ll \frac{X_j}{j} \tag{6.20}$$

for j as in (6.14).

Now we will show, that we get the same estimate for the other smaller and larger j.

6.4. **Estimating the exponential sum for long b-ary expansion.** In view of (6.14) we now concentrate on values for j satisfying (6.15).

In this case we start with the same assumptions for Δ^{-1} and H as above, *i.e.* $\Delta^{-1}, H \ll (\log N)$.. Thus for every j such that (6.16) holds we get by an application of Lemma 5.3

$$S(\boldsymbol{v},j) \ll X_j(\log X_j)^{-G}.$$

Otherwise, if (6.16) does not hold we get for every j in (6.15) together with $|b|^{\frac{j}{d}} \ll X_j \ll |b|^{\frac{J}{d}}$ that

$$0 \ll |\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{-\frac{j}{2d}} \ll |f'(z)| \ll |\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{J-j-\frac{j}{2d}} \ll |\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{-\frac{j}{2d}}\,(\log N)^{\widetilde{C_2}}. \tag{6.21}$$

Now we use the inequalities (6.21) to apply Lemma 3.6 with

$$F = \operatorname{tr}\left((\widetilde{v_1}+b\widetilde{v_2})\frac{f(z_n)}{b^{j+1}}\right),$$

$m = |\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{-\frac{j}{d}}$, and $\delta = |\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{-\frac{j}{d}}\,(\log N)^{\widetilde{C_2}}$. Thus for j as in (6.15) we get with $\sigma = 2G$ that

$$\begin{aligned}S(\boldsymbol{v},j) &\ll \frac{\sqrt{X_j}}{|\widetilde{v_1}+b\widetilde{v_2}|\,|b|^{-\frac{j}{d}}}+\frac{X_j}{(\log X_j)^{\sigma/2}}+s\left(\frac{3-\delta}{1-\delta}\right)\sqrt{X_j(\log X_j)^{\sigma}}\\ &\ll \frac{\sqrt{X_j}\,|b|^{\frac{j}{d}}}{|\widetilde{v_1}+b\widetilde{v_2}|}+\frac{X_j}{(\log X_j)^G}.\end{aligned} \tag{6.22}$$

Plugging this into (6.11) yields

(6.23)
$$\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{\left|b\right|^{2l}}\right| \ll X_j\left(\mu^k\Delta^2+\frac{2\mu^k}{H+1}+\left(\frac{\mu}{\left|b\right|^2}\right)^k\right.$$
$$\left.+\frac{\mu^k}{X_j}\left\{\sum_{\mathbf{0}<\|\boldsymbol{v}\|_\infty\leq\Delta^{-1}}+\sum_{\mathbf{0}<\|\boldsymbol{v}\|_\infty\leq H}\right\}\frac{1}{r(\boldsymbol{v})}\left(\frac{\sqrt{X_j}\left|b\right|^{\frac{j}{d}}}{\left|\widetilde{v_1}+b\widetilde{v_2}\right|}+\frac{X_j}{(\log X_j)^3}\right)\right).$$

Now we set k and H and get together with (4.4) that
$$k:=\max\left(1,\frac{\frac12\log X_j+\log 4C_\Delta^2-\frac{j}{d}\log\left|b\right|}{\log C_\mu}\right),\quad H:=\mu^k\log X_j,\quad \Delta^{-1}=\frac{\left|b\right|^k}{2c_\Delta}.$$
This yields
$$\mu^k\Delta^2=\frac{\left|b\right|^{\frac{j}{d}}}{\sqrt{X_j}},\quad \mu^k\leq\left|b\right|^{2k}\ll\left(\frac{X_j}{\left|b\right|^{\frac{2j}{d}}}\right)^{\frac{\log\left|b\right|}{\log C_\mu}},\quad \left(\frac{\mu}{\left|b\right|^2}\right)^k=\frac{1}{C_\mu^k}\ll\frac{\left|b\right|^{\frac{j}{d}}}{\sqrt{X_j}}.$$
Furthermore we get that
$$\left|\widetilde{v_1}+b\widetilde{v_2}\right|=\left|(1,b)(v_1,v_2)^t\Xi^{-1}\right|\gg\left|(v_1,v_2)^t\right|\gg\sqrt{v_1v_2}.$$
Putting all this in (6.23) yields
$$\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{\left|b\right|^{2l}}\right|\ll\sqrt{X_j}\left|b\right|^{\frac{j}{d}}+\frac{X_j}{j}+\left(\frac{X_j}{\left|b\right|^{\frac{2j}{d}}}\right)^{\frac{\log\left|b\right|}{\log C_\mu}}\left(\sqrt{X_j}\left|b\right|^{\frac{j}{d}}+X_j(\log X_j)^{-3}\right) \tag{6.24}$$
for j as in (6.15).

6.5. **Iterative estimation for short b-ary expansion.** We finally consider the case of j satisfying (6.13). This will be the hardest part as by our assumptions on H and Δ^{-1} we have
$$\left|\widetilde{v_1}+b\widetilde{v_2}\right|\ll\gg\left|b\right|^j.$$
In order to cope with this we adopt the idea of Nakai and Shiokawa [22, p.278ff] applying Proposition 5.1 iteratively. If there is no such s as assumed in that proposition, we will apply Lemma 3.6 and Lemma 3.7.

By the assumption $j\leq l+C_l\log\log N$ we get
$$\left|b\right|^j\leq(\log N)^{C_l\log\left|b\right|+o(1)}. \tag{6.25}$$
Furthermore we define g to be the polynomial
$$g(z):=\frac{\widetilde{v_1}+b\widetilde{v_2}}{b^j}f(z),$$
and β_i for $i=0,1,\dots,d$ its coefficients,
$$\beta_i=\frac{\widetilde{v_1}+b\widetilde{v_2}}{b^j}\alpha_i. \tag{6.26}$$

Now we start with the application of Proposition 5.1. We assume first that $1\leq s\leq d$. Then we set
$$H_d=H_d^*+C_1\log\left|b\right|+1,\quad H_d^*=2^{3(d+2)}+2^{d+3}G$$
and define H_r^*, H_r, and h_r $(1\leq r<d)$ inductively by
$$\begin{aligned}H_r^*&=2^{3(r+2)}+2^{r+3}(G+H_{r+1})+2r\sum_{i=r+1}^{d}H_r,\\ H_r&=H_r^*+2(C_1\log\left|b\right|+1)\quad\text{and}\\ h_r&=H_r^*+C_1\log\left|b\right|+1.\end{aligned}$$

Let j be such that $l \leq j \leq l + C_l \log\log N$ and that there are coprime pairs of Gaussian integers $(a_d, q_d), \ldots, (a_{s+1}, q_{s+1})$ such that

$$1 \leq |q_r|^2 \leq (\log X_j)^{2h_r} \quad \text{and} \quad \left|\alpha_r - \frac{a_r}{q_r}\right| \leq \frac{(\log X_j)^{h_r}}{|q_r|\, X_j^{\frac{r}{2}}} \quad (s < r \leq d),$$

but there is no pair (a_s, q_s) such that

$$1 \leq |q_s|^2 \leq (\log X_j)^{2h_s} \quad \text{and} \quad \left|\alpha_s - \frac{a_s}{q_s}\right| \leq \frac{(\log X_j)^{h_s}}{|q_s|\, X_j^{\frac{s}{2}}}.$$

We denote the set of all j with that property by $\mathbb{J}_s$.

For every $j \in \mathbb{J}_s$ we have

$$1 \leq \left|b^j q_r\right| \leq (\log X_j)^{2H_r} \quad \text{and} \quad \left|\beta_r - \frac{(\widetilde{v_1} + b\widetilde{v_2})a_r}{b^j q_r}\right| \leq \frac{(\log X_j)^{H_r}}{|b^j q_r|\, X_j^{\frac{r}{2}}}$$

for $s < r \leq d$, and, however, there is no pair of coprime Gaussian integers (A_s, Q_s) such that

$$1 \leq |Q_s| \leq (\log X_j)^{2H_s^*} \quad \text{and} \quad \left|\beta_r - \frac{A_s}{Q_s}\right| \leq \frac{(\log X_j)^{H_s^*}}{|Q_s|\, X_j^{\frac{s}{2}}},$$

since, if there were such A_s and Q_s, we would get that

$$1 \leq |(\widetilde{v_1} + b\widetilde{v_2})Q_s|^2 \leq (\log X_j)^{2H_s^*+t} \leq (\log X_j)^{2h_s}$$

and together with (6.25) that

$$\left|\alpha_s - \frac{b^j A_s}{\widetilde{v_1} + b\widetilde{v_2} Q_s}\right| \leq \frac{(\log X_j)^{H_s^* + C_1 \log|b| + 1}}{|(\widetilde{v_1} + b\widetilde{v_2})Q_s|\, X_j^{\frac{s}{2}}} \leq \frac{(\log X_j)^{h_s}}{|(\widetilde{v_1} + b\widetilde{v_2})Q_s|\, X_j^{\frac{s}{2}}},$$

which contradicts the assumption that $j \in \mathbb{J}_s$.

Thus an application of Proposition 5.1 with H_i, H_s^* and $K_i = 2H_i$, $K_i^* = 2H_i^*$ yields

$$S(\boldsymbol{v}, j) \ll X_j (\log X_j)^{-G}$$

for all $j \in \mathbb{J}_1 \cup \cdots \cup \mathbb{J}_d$.

Now we denote by $\mathbb{J}_0$ all positive integers j with $l \leq j \leq l + C_1 \log\log N$ and $j \notin \mathbb{J}_1 \cup \cdots \cup \mathbb{J}_d$. Thus it remains to estimate $S(\boldsymbol{v}, j)$ for these j. Therefore we will apply Lemma 3.6 and the Lemma 3.7.

For $j \in \mathbb{J}_0$ we get that there exist coprime pairs (a_r, q_r) of Gaussian integers such that

$$1 \leq |q_r|^2 \leq (\log X_j)^{2h_r} \quad \text{and} \quad \left|\alpha_r - \frac{a_r}{q_r}\right| \leq \frac{(\log X_j)^{h_r}}{|q_r|\, X_j^{\frac{r}{2}}} \quad (1 \leq r \leq d).$$

We set $\Omega_r = \alpha_r - \frac{a_r}{q_r}$ for $r = 1, \ldots, d$. Furthermore we denote by a the greatest common divisor of $a_1, \ldots, a_d$ and by q the least common multiple of $q_1, \ldots, q_d$. Furthermore we define c_r by

$$\frac{a_r}{q_r} = \frac{a}{q} c_r \quad (r = 1, \ldots, d).$$

Then we can rewrite the exponential sum as follows:

$$\begin{aligned} S(\boldsymbol{v}, j) &= \sum_{n \in I_j} e\left(\operatorname{tr}\left((\widetilde{v_1} + b\widetilde{v_2})\frac{f(z_n)}{b^{j+1}}\right)\right) \\ &= \sum_{\lambda \in r(b^{j+1}q)} e\left(\operatorname{tr}\left(\frac{\hat{v}a}{b^{j+1}q}\sum_{k=1}^{d} c_k \lambda^k\right)\right) \sum_{\substack{\mu \\ \exists n \in I_j : \mu q + \lambda = z_n}} e\left(\operatorname{tr}\left(\frac{\hat{v}}{b^{j+1}}\sum_{k=1}^{d} \Omega_k (\mu q + \lambda)^k\right)\right) \end{aligned}$$

where $r(b^{j+1}q)$ denotes a complete system of residues modulo $b^{j+1}q$ and $\hat{v} := \widetilde{v_1} + b\widetilde{v_2}$.

We first consider the second sum. Let $R_0 = \mathbb{Z}[i] \cap (b^{j+1}q) \cdot \{\alpha + \beta i : 0 \leq \alpha, \beta \leq 1\}$ and let T_0 the set of translates such that R_0 tiles $\mathbb{Z}^2$. Furthermore we set T the set of all $t \in T_0$ that do not have empty intersection with I_j, thus

$$T := \{t \in T_0 : (R_0 + t) \cap \{z_n : n \in I_j\} \neq \emptyset\}. \tag{6.27}$$

Then it is clear that $|T| \ll X_j \left|b^{j+1}q\right|^{-2}$. Furthermore let $\mathcal{T}$ denote the area covered by the translates of T, *i.e.*,

$$\mathcal{T} := \bigcup_{t \in T} (R_0 + t).$$

Thus we fix a $\lambda \in R_0$ and get that

$$\sum_{\substack{\mu \\ \exists n \in I_j : \mu q + \lambda = z_n}} e\left(\operatorname{tr}\left(\frac{\hat{v}}{b^{j+1}} \sum_{k=1}^{d} \Omega_k (\mu q + \lambda)^k\right)\right) \leq \sum_{\mu \in T} e\left(\operatorname{tr}\left(\frac{\hat{v}}{b^{j+1}} \sum_{k=1}^{d} \Omega_k (\mu q + \lambda)^k\right)\right).$$

Now we want to apply Lemma 3.5 together with the idea in the proof of Lemma 3.6. Therefore we set

$$F_\lambda(x, y) := e\left(\operatorname{tr}\left(\frac{\hat{v}}{b^{j+1}} \sum_{k=1}^{d} \Omega_k ((x + iy)q + \lambda)^k\right)\right).$$

Then we get for the derivatives

$$\frac{\partial F_\lambda(x,y)}{\partial x} \ll\gg \frac{\partial F_\lambda(x,y)}{\partial y} \ll \frac{\hat{v}}{|b|^j} \sum_{k=1}^{d} k\,|q|\, \frac{(\log X_j)^{H_k}}{q_k X_j^{k/2}} X_j^{\frac{k-1}{2}} \ll \frac{\hat{v}}{|b|^j} X_j^{-\frac{1}{2}}\,|q|\,(\log X_j)^{H_1^*}.$$

As in the proof of Lemma 3.6 we first consider a single square. We denote by $D_\nu := \{z = x + iy \in \mathbb{Z}[i] : -\nu \leq x, y \leq \nu\}$. Thus an application of Lemma 3.5 yields

$$\sum_{x+iy \in D_\nu} F_\lambda(x,y) = \sum_{x=-\nu}^{\nu} \sum_{y=-\nu}^{\nu} F_\lambda(x,y) = \int_{-\nu}^{\nu} \int_{-\nu}^{\nu} F_\lambda(x,y) \mathrm{d}x\mathrm{d}y + \mathcal{O}(\nu).$$

Now we again want to split $\mathcal{T}$ up into squares. Therefore we note that we had assumed that $|I_j| = X_j$ and thus we can consider I_j as an annulus, *i.e.* as set $\{z \in \mathbb{C} : M_j \leq |z|^2 < M\}$. Thus we choose a $\sigma > 0$ and tessellate $\mathcal{T}$ by squares of side length $\sqrt{|T| / (\log |T|)^\sigma}$. Then we can glue all squares in the interior of $\mathcal{T}$ together and estimating their contribution on the boundary to the error term. Thus we get

$$\sum_{x+iy \in T} F_\lambda(x,y) = \iint_{\mathcal{T}} F_\lambda(x,y) \mathrm{d}x\mathrm{d}y + \mathcal{O}\left(\frac{|T|}{(\log |T|)^{\sigma/2}}\right).$$

Putting everything together yields

$$\begin{aligned}
S(\boldsymbol{v}, j) &= \sum_{n \in I_j} e\left(\operatorname{tr}\left((\widetilde{v_1} + b\widetilde{v_2}) \frac{f(z_n)}{b^{j+1}}\right)\right) \\
&= \sum_{\lambda \in r(b^j q)} e\left(\operatorname{tr}\left(\frac{\nu a}{b^j q} \sum_{k=1}^{d} c_k \lambda^k\right)\right) \left\{\iint_{\mathcal{T}} F_\lambda(x,y) \mathrm{d}x\mathrm{d}y + \mathcal{O}\left(\frac{|T|}{(\log |T|)^{\sigma/2}}\right)\right\} \\
&= \sum_{\lambda \in r(b^j q)} e\left(\operatorname{tr}\left(\frac{\nu a}{b^j q} \sum_{k=1}^{d} c_k \lambda^k\right)\right) \frac{1}{\left|b^{j+1}q\right|^2} \iint_{M_j \leq |z|^2 < M} G(z) \mathrm{d}z + \mathcal{O}\left(\frac{X_j}{(\log X_j)^\sigma}\right),
\end{aligned}$$

where

$$G(z) := e\left(\operatorname{tr}\left(\frac{\hat{v}}{b^{j+1}} \sum_{k=1}^{d} \Omega_k z^k\right)\right).$$

Finally we define rationals $R_i/Q \in \mathbb{Q}(i)$ for $i = 1, \ldots, d$ by

$$\frac{R_i}{Q} = \frac{\hat{v}}{b^j} \frac{a c_i}{q}.$$

Thus estimating the integral trivially and noting that

$$N(\hat{v}Q) = N(b^{j+1}R_i q_i/a_i) \ll\gg N(b^{j+1}R_i\alpha_i^{-1}) \ll\gg N(b^{j+1}R_i) \gg N(b^{j+1})$$

we get by an application of Lemma 3.7

$$\begin{aligned}(6.28)\quad S(\boldsymbol{v}, j) &= \sum_{n\in I_j} e\left(\operatorname{tr}\left((\widetilde{v_1} + b\widetilde{v_2})\frac{f(z_n)}{b^{j+1}}\right)\right) \\ &\ll \frac{\left|b^j q\right|^2}{N(Q)}(N(Q))^{1-\frac{1}{d}+\varepsilon}\frac{X_j}{\left|b^j q\right|^2} + \frac{X_j}{(\log X_j)^\sigma} \\ &\ll X_j\left((N(\hat{v}^{-1}b^{j+1}))^{-\frac{1}{d}+\varepsilon} + (\log X_j)^{-\sigma}\right).\end{aligned}$$

Plugging this into (6.11) yields

(6.29)

$$\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right) - \frac{X_j}{|b|^{2l}}\right| \ll X_j\left(\mu^k\Delta^2 + \frac{2\mu^k}{H+1} + \left(\frac{\mu}{|b|^2}\right)^k + \mu^k\left\{\sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\Delta^{-1}} + \sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le H}\right\}\frac{1}{r(\boldsymbol{v})}\left((N(\hat{v}^{-1}b^{j+1}))^{-\frac{1}{d}+\varepsilon} + (\log X_j)^{-\sigma}\right)\right).$$

Now we set σ, k, and H with the same values as in (6.19) and get together with (4.4) that

$$\sigma := G,\quad k := C_k\log\log X_j,\quad H := \mu^k\log X_j,\quad \Delta^{-1} = \frac{(\log X_j)^{C_k\log|b|}}{2c_\Delta},$$

for C_k an arbitrary constant.

We note that

$$|\widetilde{v_1} + b\widetilde{v_2}| = \left|(1,b)(v_1,v_2)^t\Xi^{-1}\right| \ll |(v_1,v_2)| \ll r(\boldsymbol{v}).$$

At this point we have to distinguish two cases according to the size of d.

- $d = 1$: By noting that $\Delta^{-1}, H \ll (\log N)$ we get that

$$\sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\log N}\frac{1}{r(\boldsymbol{v})}(N(\hat{v}^{-1}b^{j+1}))^{-1+\varepsilon} \ll \sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\log N}\frac{|\widetilde{v_1}+b\widetilde{v_2}|}{|b|^{(2-\varepsilon)\frac{j+1}{d}}} \ll \frac{(\log N)^4}{|b|^{\frac{2j}{d}}}.$$

- $d \geq 2$: In this case get that

$$r(\boldsymbol{v})^{-1} \ll |\widetilde{v_1}+b\widetilde{v_2}|^{-1} \ll |\widetilde{v_1}+b\widetilde{v_2}|^{-\frac{2}{d}}.$$

  This together with $\Delta^{-1}, H \ll \log N$ yields

$$\sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\log N}\frac{1}{r(\boldsymbol{v})}(N(\hat{v}^{-1}b^{j+1}))^{-\frac{1}{d}+\varepsilon} \ll \sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\log N}\frac{1}{|b|^{(2-\varepsilon)\frac{j+1}{d}}} \ll \frac{(\log N)^2}{|b|^{\frac{2j}{d}}}.$$

Therefore we get in any case that

$$\sum_{\boldsymbol{0}<\|\boldsymbol{v}\|_\infty\le\log N}\frac{1}{r(\boldsymbol{v})}(N(\hat{v}^{-1}b^{j+1}))^{-\frac{1}{d}+\varepsilon} \ll \frac{(\log N)^4}{|b|^{\frac{2j}{d}}}.$$

Putting this all in (6.29) yields

$$(6.30)\quad \left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right) - \frac{X_j}{|b|^{2l}}\right| \ll X_j\left((\log X_j)^{-1} + \frac{(\log N)^4\log X_j}{|b|^{\frac{2j}{d}}}\right) \ll \frac{X_j}{j} + X_j\frac{(\log N)^5}{|b|^{\frac{2j}{d}}}.$$

6.6. **Putting all together.** Now we have reached the final state of the proof. In order to finish we will put (6.20), (6.24), and (6.30) together and consider the corresponding intervals, which are described in (6.14), (6.15), and (6.13), respectively. Thus

$$\sum_{l\leq j\leq J}\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{|b|^{2l}}\right|\ll S_1+S_2+S_3, \tag{6.31}$$

where

$$S_1=\sum_{l\leq j\leq J}\frac{X_j}{j},$$

$$S_2=\sum_{l\leq j\leq l+C_l\log\log N}X_j\frac{(\log N)^5}{|b|^{\frac{2j}{d}}},$$

$$S_3=\sum_{J-C_u\log\log N\leq j\leq J}\sqrt{X_j}\,|b|^{\frac{j}{d}}+\left(\frac{X_j}{|b|^{\frac{2j}{d}}}\right)^{\frac{\log|b|}{\log C_\mu}}\left(\sqrt{X_j}\,|b|^{\frac{j}{d}}+X_j(\log X_j)^{-3}\right).$$

We estimate each sum and easily get for the first one

$$S_1\ll M.$$

The second one is a bit more delicate and simplifies to

$$S_2\ll\sum_{l\leq j\leq l+C_l\log\log N}M\frac{(\log N)^5}{|b|^{\frac{2j}{d}}}\ll M\frac{(\log N)^5}{|b|^{\frac{2}{d}(C_l\log\log N)}}\ll M,$$

where we have assumed that $C_l\geq 5$. For the third sum we have to do a little more work to get

$$\begin{aligned}S_3&\ll\sum_{J-C_u\log\log N\leq j\leq J}\sqrt{M}\,|b|^{\frac{j}{d}}+\left(\frac{M}{|b|^{\frac{2j}{d}}}\right)^{\frac{\log|b|}{\log C_\mu}}\left(\sqrt{M}\,|b|^{\frac{j}{d}}+M\right)\\&\ll\sqrt{M}\,|b|^{\frac{J}{d}}+\left(\frac{M}{|b|^{\frac{2J}{d}}}\right)^{\frac{\log|b|}{\log C_\mu}}\left(\sqrt{M}\,|b|^{\frac{J}{d}}+M\right)\\&\ll M.\end{aligned}$$

Putting this in (6.31) yields

$$\sum_{l\leq j\leq J}\left|\sum_{n\in I_j}\mathcal{I}\left(\frac{f(z_n)}{b^{j+1}}\right)-\frac{X_j}{|b|^{2l}}\right|\ll M\ll\frac{N}{\log N}$$

and the main theorem is proven.

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