The Northwest corner rule revisited

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Abstract

Under which conditions can one permute the rows and columns in an instance of the transportation problem, such that the Northwest corner rule solves the resulting permuted instance to optimality? And under which conditions will the Northwest corner rule find the optimal solution for every possible permutation of the instance?

We show that the first question touches the area of NP-completeness, and we answer the second question by a simple characterization of such instances.

Keywords: Transportation problem; basic feasible solution; computational complexity; sum matrix.

1 Introduction

The transportation problem is an important and well-investigated problem in operations research: There are m sources $s_1, \ldots, s_m$ with a supply of $a_i > 0$ units at the ith source ($i = 1, \ldots, m$), and there are n sinks $t_1, \ldots, t_n$ with a demand of $b_j > 0$ units at the jth sink ($j = 1, \ldots, n$). These supplies and demands satisfy $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. The cost for transporting one unit from the ith source to the jth sink is $c_{i,j} \geq 0$. The $m \times n$ cost matrix $C = (c_{i,j})$, the m-dimensional supply vector $a = (a_i)$, and the n-dimensional demand vector $b = (b_j)$ form an instance of the transportation problem. The goal is to find a transportation plan that satisfies all the demand and that minimizes the overall transportation cost:

$$\begin{array}{lll} \min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j}\, x_{i,j} & \\ \text{s.t.} & \sum_{j=1}^{n} x_{i,j} = a_i & \text{for } i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{i,j} = b_j & \text{for } j = 1, \ldots, n \\ & x_{i,j} \geq 0 & \text{for } i = 1, \ldots, m,\ j = 1, \ldots, n. \end{array}$$

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1. Initialize the Northwest corner (i,j) with $i := 1$ and $j := 1$.
2. Send as many units as possible from i to j by setting $x_{i,j} := \min\{a_i, b_j\}$.
3. Adjust the supply $a_i := a_i - x_{i,j}$ and the demand $b_j := b_j - x_{i,j}$.
   If $a_i = 0$ then $i := i+1$, and if $b_j = 0$ then $j := j+1$.
4. If there still is unsatisfied demand, go back to Step 2.

Figure 1: The Northwest corner rule.

Here $x_{i,j}$ denotes the quantity shipped from source i to sink j. We refer the reader to the book by Ahuja, Magnanti & Orlin [1] for a wealth of information on the transportation problem and its applications.

The co-called *Northwest corner rule* appears in virtually every text-book chapter on the transportation problem. It is a standard method for computing a basic feasible solution (which will be denoted $\mathrm{BFS}^{\mathrm{NW}}$ in the following), and it does so by fixing the values of the basic variables one by one and starting from the Northwest corner of matrix C; see Figure 1 for a short description.

Since the Northwest corner rule does not even look at the cost matrix C, the objective value of $\mathrm{BFS}^{\mathrm{NW}}$ can be very bad. Thus the solution $\mathrm{BFS}^{\mathrm{NW}}$ usually just serves as a starting point for the simplex algorithm or for some other LP solving approach. But sometimes we are lucky and it happens that the starting solution $\mathrm{BFS}^{\mathrm{NW}}$ itself is an optimal solution, in which case the simplex algorithm terminates right away. That is, for instance, the case whenever the cost matrix C is a *Monge matrix* (see Hoffman [6], and Burkard, Klinz & Rudolf [3]). Are there other (non-Monge) cases where the Northwest corner rule hits the optimal solution? The answer is yes. And the combinatorial structure of these YES-cases can be quite chaotic, as demonstrated by the following example.

Example 1.1 *We apply the Northwest corner rule to an arbitrary cost matrix. Then we raise all costs that are not used by* BFS^{NW} *to sufficiently large values, so that* BFS^{NW} *becomes an optimal solution. The resulting cost matrix can almost be arbitrary, and it carries very little combinatorial structure.*

The following facts are straightforward: By renumbering the sources and the sinks in the transportation problem, one does not change the optimal objective value; the renumbering just yields an equivalent *permuted* instance. But by renumbering the sources and the sinks in the transportation problem, one may drastically change the behavior of the Northwest corner rule.

Is it always possible to renumber an instance, such that the resulting permuted instance is solved to optimality by the Northwest corner rule? The following example provides a negative answer.

Example 1.2 *Consider an instance with three sources s_1, s_2, s_3 and four sinks $t_1, \ldots, t_4$. Every source has a supply of 2, sinks t_1, t_2, t_3 have a demand of 1, and sink t_4 has a demand of 3. Transportation is cheap between s_i and t_i for $1 \le i \le 3$, is cheap between s_i and t_4 for $1 \le i \le 3$, and is expensive otherwise; see Figure 2.*

The optimal solution has cost 0; it sends one unit from s_i to t_i for $1 \le i \le 3$, and one unit from s_i to t_4 for $1 \le i \le 3$. It is easily verified that there is no way of permuting the rows and columns such that the Northwest corner rule would find a solution of cost 0.

$a \backslash b$	1	1	1	3
2	0	1	1	0
2	1	0	1	0
2	1	1	0	0

Figure 2: Costs, supplies, and demands for the instance discussed in Example 1.2.

Actually, if there are (1) at most two sources, or if there are (2) at most two sinks, or if there are (3) exactly three sources and exactly three sinks, then the instance can always be permuted so that the Northwest corner rule detects an optimal solution. For all other combinations of m and n, there exist instances for which no such permutation is possible. How hard is it to recognize such an instance?

Problem: Good Permutation

Instance: An instance (C, a, b) of the transportation problem.

Question: Can the instance be permuted, so that the Northwest corner rule finds an optimal solution?

We will show in Section 2 that problem Good Permutation is NP-complete, which means that its combinatorial behavior is quite messy. How hard is it to recognize the diametrical type of instance? How hard is it to recognize instances that do not possess a single row/column permutation that could prevent the Northwest corner rule from hitting an optimal solution?

Problem: Bad Permutation

Instance: An instance (C, a, b) of the transportation problem.

Question: Can the instance be permuted, so that the Northwest corner rule does not find an optimal solution?

We provide a complete answer to this question in Section 3: If the cost matrix C is a sum matrix, then all feasible solutions have the same objective value; hence in such a case the answer to Bad Permutation must be negative. It turns out that in all remaining cases, the answer to Bad Permutation is positive. Note that the exact values of the supplies and demands do not play any role in this. Since sum matrices are straightforward to recognize, this yields that problem Bad Permutation can be solved in polynomial time.

2 The intractability result

In this section we discuss the behavior of the Good Permutation problem. We start with some technical preliminaries.

We recall that a *caterpillar* graph is a tree that turns into a path (the so-called *spine* of the caterpillar) if all its leaves are removed. An equivalent definition (see Arnborg, Proskurowski & Seese [2]) is that caterpillars are the connected graphs of path-width one. Yet another equivalent characterization says that caterpillars are precisely those trees that do not contain the forbidden seven-vertex sub-tree T_7 that results from the star $K_{1,3}$ by subdividing every edge. Note that in Example 1.2 the edges of cost 0 induce this forbidden sub-tree T_7 with the three subdividing vertices in the three sources.

Every basic feasible solution of the transportation problem has $m + n - 1$ variables $x_{i,j}$ in the basis, and if it is non-degenerate then all these variables take non-zero values. Every basic feasible solution BFS corresponds to a graph $G(\mathrm{BFS})$ whose vertex set are the sources and sinks, and whose edges connect a source i to a sink j if and only if $x_{i,j}$ is in the basis; in fact $G(\mathrm{BFS})$ is always a tree. The following observation is folklore.

Observation 2.1 *Let* BFS *be a basic feasible solution for an instance of the transportation problem. Then the instance can be permuted so that the Northwest corner rule computes* BFS *if and only if the graph* $G(\mathrm{BFS})$ *is a caterpillar graph.*

We now turn to the NP-completeness proof for the Good Permutation problem. Our reduction is done from the following version of the Hamiltonian path problem in bipartite graphs; see Garey & Johnson [4].

> Problem: Hamiltonian Path
>
> Instance: A bipartite graph $G = (X \cup Y, E)$ with bipartition $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, and with edge set $E \subseteq X \times Y$.
>
> Question: Does G have a Hamiltonian path that starts in x_1 and ends in some vertex in Y?

Consider an instance of Hamiltonian Path with $k \geq 10$. We construct the following instance of the transportation problem with $m = 2k$ sources and $n = 2k$ sinks:

- For every vertex $x \in X$, we create a corresponding main source $s(x)$ and a corresponding dummy sink $t'(x)$. For every vertex $y \in Y$, we create a corresponding main sink $t(y)$ and a corresponding dummy source $s'(y)$.

- The main source $s(x_1)$ that corresponds to vertex x_1 has supply $k^2 - k + 2$, and the remaining $k-1$ main sources all have supply k^2+2. All dummy sources have supply 1. All main sinks have demand k^2+1, and all dummy sinks have demand 1.
- For every vertex $x \in X$ the transportation cost between $s(x)$ and $t'(x)$ is 0, and for every vertex $y \in Y$ the transportation cost between $s'(y)$ and $t(y)$ is 0. All the other transportation costs between main sources and dummy sinks, between dummy sources and main sinks, and between dummy sources and dummy sinks are 1.
  Whenever $[x, y] \in E$ then the cost between main source $s(x)$ and main sink $t(y)$ is 0. All other costs between main sources and main sinks are 1.

This completes the construction. The correctness argument in the following paragraphs is slightly non-standard, since it does not directly establish a bijection between the YES-instances of the two problems, but is also built around the optimal objective value of the transportation instance: We will show that the instance of Hamiltonian Path has answer YES, if and only if the instance of the transportation problem (i) has optimal objective value 0 and (ii) forms a YES-instance of the Good Permutation problem.

Lemma 2.2 *If the constructed instance of the transportation problem has optimal objective value 0 and forms a YES-instance of* Good Permutation*, then the underlying instance of* Hamiltonian Path *has answer YES.*

Proof. Observation 2.1 yields that there is a basic feasible solution BFS of cost 0, whose graph $G(\text{BFS})$ is a caterpillar. Our first goal is to show that every edge in $G(\text{BFS})$ has cost 0. Suppose otherwise, and consider a connected component G^* of the graph that results from $G(\text{BFS})$ by removing all edges with strictly positive cost. Then G^* itself is a caterpillar. Since at cost 0 a dummy sink $t'(x)$ can only receive units from the main source $s(x)$ that belongs to the same vertex $x \in X$, component G^* contains either both of $s(x)$ and $t'(x)$ or neither of them. Symmetrically, for every vertex $y \in Y$ component G^* contains either both $s'(y)$ and $t(y)$ or neither of them.

Let $X^* \subseteq X$ and $Y^* \subseteq Y$ respectively denote the sets of vertices for which G^* contains both corresponding (source and sink) vertices. Furthermore let $\alpha = |X^*|$ and $\beta = |Y^*|$, and note that $0 \le \alpha, \beta \le k$. The total supply in the sources in G^* equals the total demand in the sinks of G^*. In case $x_1 \notin X^*$ holds, this equality yields

$$(k^2+3)\alpha \;=\; (k^2+2)\beta. \qquad (1)$$

Since the numbers k^2+2 and k^2+3 are relative prime, (1) implies that α is a multiple of k^2+2 and that β is a multiple of k^2+3. From $0 \le \alpha, \beta \le k$ we then get $\alpha = \beta = 0$, which is impossible. In the only remaining case we have $x_1 \in X^*$, which implies

$$(k^2-k+3) + (k^2+3)(\alpha-1) \;=\; (k^2+2)\beta. \qquad (2)$$

By rewriting (2) in the form $k-\alpha = (k^2+2)(\alpha-\beta)$, we see that $k-\alpha$ (which lies between 0 and k) must be a multiple of k^2+2. This yields $\alpha = k$ and $\beta = k$, and hence the component G^* coincides with $G(\mathrm{BFS})$. Consequently, every edge in $G(\mathrm{BFS})$ has cost 0.

Since every dummy source and every dummy sink have only a single incident edge of cost 0, they form $2k$ leaves in the caterpillar $G(\mathrm{BFS})$. Their neighbors are the main sources and main sinks, which hence must all lie on the spine of the caterpillar. Since the spine alternately visits main sources and main sinks, one of its endpoints is a main source $s(x_i)$ next to some main sink $t(y_j)$. Then main source $s(x_i)$ and dummy source $s'(y_j)$ can only send their supply to dummy sink $t'(x_i)$ (with demand 1) and to main sink $t(y_j)$ (with demand k^2+1). This yields that the supply at $s(x_i)$ is at most k^2+1, which in turn yields $i = 1$. All in all, this shows that the spine induces a Hamiltonian path in the underlying bipartite graph that has vertex x_1 as an endpoint. □

Lemma 2.3 *If the underlying instance of* HAMILTONIAN PATH *has answer YES, then the constructed instance of the transportation problem has optimal objective value 0 and forms a YES-instance of* GOOD PERMUTATION.

Proof. Without loss of generality we assume that the Hamiltonian path visits the vertices in $X \cup Y$ in the order

$$x_1, y_1,\ x_2, y_2,\ x_3, y_3,\ \ldots\ldots,\ x_{k-1}, y_{k-1},\ x_k, y_k.$$

We define a basic feasible solution BFS: For $i = 1, \ldots, k-1$ we send k^2-k+i units from main source $s(x_i)$ to main sink $t(y_i)$, and $k-i$ units from main source $s(x_{i+1})$ to main sink $t(y_i)$. For $i = 1, \ldots, k$ we send 1 unit from main source $s(x_i)$ to the corresponding dummy sink $t'(x_i)$, and 1 unit from every dummy source $s'(y_i)$ to the corresponding main sink $t(y_i)$.

This defines a basic feasible solution BFS of cost 0 (which of course is optimal). The underlying graph $G(\mathrm{BFS})$ is a caterpillar whose spine path runs through the main sources and main sinks corresponding to the vertices along the Hamiltonian path; the leaves in $G(\mathrm{BFS})$ are the dummy sources and dummy sinks. Finally Observation 2.1 completes the argument. □

Since the optimal objective value for the transportation problem can be computed in polynomial time, the combination of Lemma 2.2 and Lemma 2.3 yields the following theorem.

Theorem 2.4 *Problem* GOOD PERMUTATION *is NP-complete.* □

3 The connection to sum matrices

In this section we discuss the behavior of the BAD PERMUTATION problem. We start with several technical definitions and observations. Unless stated otherwise, all matrices in this section have m rows and n columns. Since the cases with a single source or single sink are trivial, we will throughout assume that $m, n \geq 2$. We stress that all supplies and demands are positive. Recall that a matrix S is a *sum matrix*, if there exist two vectors $u = (u_i)$ and $v = (v_j)$ such that $s_{i,j} = u_i + v_j$ for all i, j.

Observation 3.1 *If C is a sum matrix, then any instance (C, a, b) of the transportation problem forms a NO-instance of problem* Bad Permutation.

Proof. If $c_{i,j} = u_i + v_j$ for all i, j, then the cost of any feasible solution $x_{i,j}$ is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{i,j}\, x_{i,j} \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} (u_i + v_j)\, x_{i,j} \;=\; \sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j.$$

Hence any feasible solution, and in particular any solution computed by the Northwest corner rule under any permutation of rows and columns, is optimal. □

The argument in Observation 3.1 also shows that for a sum matrix S, the two instances (C, a, b) and $(C + S, a, b)$ of the transportation problem have the same set of optimal solutions. Furthermore, the Northwest corner rule outputs for both instances the same feasible solution. Thus for our purposes instance (C, a, b) and instance $(C + S, a, b)$ are equivalent.

For any instance $I = (C, a, b)$ of the transportation problem, the *transposed* instance $I^T = (C^T, b, a)$ has the transposed matrix C^T as cost matrix, and essentially switches the roles of sources and sinks such that all shipments flow into the opposite direction. Then I and I^T have the same optimal objective value. Furthermore, if we apply the Northwest corner rule to instance I^T, then it finds a transposed copy of the feasible solution that it determines for instance I. So for our purposes these two instances are equivalent.

Observation 3.2 *Let C be a matrix that is not a sum matrix. Then there exists a row $p \neq 1$, two columns q and r with $q \neq r$, and a sum matrix S, such that all entries in the matrix $C' := C + S$ are non-negative real numbers, such that $c'_{1,j} = 0$ for $j = 1, \ldots, n$, and such that $c'_{p,q} \neq c'_{p,r}$.*

Proof. First suppose for the sake of contradiction that all rows $p \neq 1$ and all columns $q \neq r$ satisfy $c_{p,q} - c_{1,q} = c_{p,r} - c_{1,r}$. Then with $u_i = c_{i,1}$ for $i = 1, \ldots, m$ and with $v_j = c_{1,j} - c_{1,1}$ for $q = 1, \ldots, n$, we arrive at the contradiction that $c_{i,j} = u_p + v_q$ is a sum matrix. Hence there are $p \neq 1$ and $q \neq r$ with $c_{p,q} - c_{1,q} \neq c_{p,r} - c_{1,r}$. Let $u_1 = 0$, let $u_i = \max_j c_{1,j}$ for $i = 2, \ldots, m$, and let $v_j = -c_{1,j}$ for $j = 1, \ldots, n$. Then the sum S matrix defined by $s_{i,j} = u_i + v_j$ has all desired properties. □

The following lemma proves the matching reverse statement for Observation 3.1.

Lemma 3.3 *If C is not a sum matrix, then any instance $I = (C, a, b)$ of the transportation problem forms a YES-instance of problem* Bad Permutation.

Proof. We assume (by transposing the entire instance if necessary) that the smallest value among all supplies $a_1, \ldots, a_m$ and all demands $b_1, \ldots, b_n$ is the demand b_1 in column 1. Furthermore we assume (by Observation 3.2 and by permuting rows) that in the first row $c_{1,j} = 0$ holds for all j, and that there exist columns q and r such that $c_{2,q} \neq c_{2,r}$ holds in the second row.

Our first goal is to construct a subset J of the columns (sinks) of cardinality $t \geq 2$, and an ordering $j(1), j(2), \ldots, j(t)$ of these t columns that has three crucial properties. The first crucial property is that the corresponding demands satisfy

$$\sum_{j \in J} b_j \;\geq\; a_1 + a_2 \;>\; (\sum_{j \in J} b_j) - b_{j(t)}. \tag{3}$$

The second crucial property is that the underlying costs $c_{2,j}$ with $j \in J$ take at least two different values. The third crucial property is that under the ordering $j(1), j(2), \ldots, j(t)$ the corresponding cost coefficients $c_{2,j(k)}$ in the second row are either in non-decreasing or in non-increasing order. Subsets of the columns that satisfy the left inequality in (3) are called *heavy*, and subsets that satisfy the second crucial property are called *mixed*.

We start our construction with the set $\{1, \ldots, n\}$ of all columns which is mixed and heavy. We repeatedly remove some column $j \neq 1$ from this set as long as the resulting set still is mixed and heavy. When no further removal is possible the process terminates, the resulting column set is the desired set J with cardinality $t \geq 2$; note that $1 \in J$. Let $j(1), j(2), \ldots, j(t)$ be an enumeration of the columns in J such that

$$c_{2,j(1)} \;\leq\; c_{2,j(2)} \;\leq\; \cdots\cdots \;\leq\; c_{2,j(t)}. \tag{4}$$

Then $t \geq 2$ implies $j(1) \neq 1$ or $j(t) \neq 1$. We only discuss the case where $j(t) \neq 1$; the other case can be handled by an analogous symmetric argument. Now why did the removal process decide to keep this last column $j(t) \neq 1$ in J? One possible reason is that $J' = J - \{j(t)\}$ is not heavy. But then we are done, as set J with the ordering in (4) possesses all three crucial properties. Note that this covers all cases with $t = 2$: These cases have $j(1) = 1$, and then our assumption $b_1 \leq a_1$ and $b_1 \leq a_2$ yields $b_1 < a_1 + a_2$, which implies that J' is not heavy. From now on we will hence assume $t \geq 3$. The other possible reason is that set J' is heavy, but not mixed. This yields

$$c_{2,j(1)} \;=\; c_{2,j(2)} \;=\; \cdots\cdots \;=\; c_{2,j(t-1)} \;<\; c_{2,j(t)}.$$

In addition, for every k with $1 \leq k \leq t-1$ and $j(k) \neq 1$ we get that $J - \{j(k)\}$ is not heavy. Since $t \geq 3$ holds, the set J with the following ordering has all three crucial properties: The ordering starts with column $j(t)$, followed by column 1, followed by the remaining $t-2$ columns in arbitrary order. This completes the construction of J and its ordering.

Our next goal is to find a permutation of instance I for which the Northwest corner does not find the optimal solution. We rename the sinks so that the columns in J in the ordering $j(1), j(2), \ldots, j(t)$ become the first t columns. We assume that the third crucial property has the corresponding costs $c_{2,1} \le c_{2,2,} \le \cdots \le c_{2,t}$ in non-decreasing order (the non-increasing case can be settled by a symmetric argument). The resulting instance is called I^+. We then create from I^+ another instance I^- by switching the positions of sources 1 and 2 in the first two rows.

How does the Northwest corner rule behave on instance I^+? It first assigns the supplies from source 1 to the first few sinks at a total cost of 0. Then it moves to the second row, and step by step assigns the supplies from source 2 to the next few sinks. By (3), there still will be unused supply at source 2 after the Northwest corner rule has served sink $t-1$. Thus it also serves sink t from source 2, then by (3) jumps to the third row at variable $x_{3,t}$, and handles the rest of the instance (this also covers the special case, where the rest of the instance is empty and the third row does not exist). And how does the Northwest corner rule behave on instance I^-? Since source 1 and source 2 have switched places, first the supply of source 2 is assigned to the first few sinks, and then the supply of source 1 is assigned to the next few sinks (at cost 0). Eventually, the Northwest corner rule assigns the last supply units from source 1 to sink t. Then it jumps to the third row at variable $x_{3,t}$, and handles the rest of instance I^- in exactly the same fashion as it did with the rest of instance I^+.

So between the two instances, the only difference in cost arises from assigning the units from source 2. For I^+ these units go to the most expensive sinks in J, whereas for I^- these units go to the cheapest sinks in J. Since by the second crucial property the costs $c_{2,j}$ with $j \in J$ take at least two different values, we get two different objective values for $\mathrm{BFS}^{\mathrm{NW}}$ on I^+ and on I^-. Hence for the permutation I^+ of I, the Northwest corner does not find an optimal solution. $\Box$

To summarize our findings in this section: An instance of problem Bad Permutation has answer NO if and only if the underlying cost matrix is a sum matrix.

Theorem 3.4 *Problem* Bad Permutation *is solvable in polynomial time.* $\Box$

4 Conclusions

We discussed permutations of the rows and columns in an instance of the transportation problem that make the Northwest corner rule perform well. We showed that recognizing the existence of such a good permutation is computationally intractable, and we characterized the instances for which all permutations are good.

Gilmore, Lawler & Shmoys [5] characterize distance matrices for the traveling salesman problem under which all traveling salesman tours have the same length. It turns out that the class of such distance matrices is exactly the class of sum matrices. The characterization in [5] and our characterization in Section 3 have a similar flavor, but the two statements seem to be independent of each other. It would be interesting to unravel a deeper connection between the two results (if such a connection indeed exists).

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