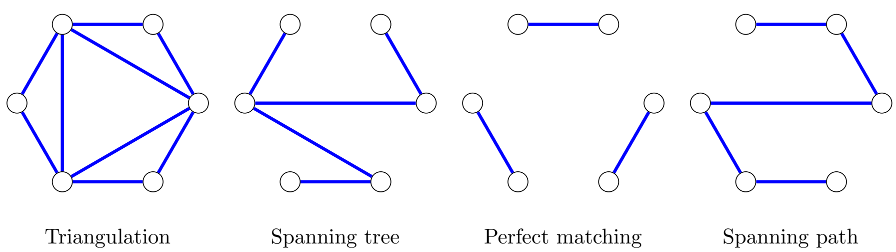
TU Graz, doc.funds Oswin Aichholzer, Cesar Ceballos

Reconfiguration of Plane Spanning Graphs

Short Abstract This project concerns combinatorial problems at the interplay between graphs and discrete structures in the areas of discrete and computational geometry, with a special focus on reconfiguration problems. Reconfiguration is the process of changing a structure into another - either through continuous motion or through discrete changes. In mathematics, the topic has a long history, ranging from knot theory via the study of bounds on the simplex method for linear programming to reconfiguration problems on graphs. Examples include solving the Rubik's cube or changing one colouring of a graph into another. In this project we concentrate on plane graphs and discrete reconfiguration steps of bounded complexity, like exchanging one edge of the graph for another edge. This operation is usually called a flip, and the flip graph is defined as the graph having a vertex for each configuration and an edge for each flip. Three questions are central: studying the connectivity of the flip graph, its diameter, and the complexity of finding the shortest flip sequence between two given configurations. There exist several classic and new results, as well as long-standing and interesting new open problems in this area.

Topics and Goals Reconfiguration problems on graphs are of fundamental interest in computational geometry, and graphs arising from finite point configurations provide a rich source of examples for such problems. Given a set S of n points in the plane in general position, one can consider the set $\mathcal{G}(S)$ of all plane (geometric) graphs on S of a certain type, for example triangulations, spanning trees, perfect matchings, spanning cycles, or spanning paths.



The elements of $\mathcal{G}(S)$ can be connected by small transformations called flips, which exchange one edge (or a constant number of edges) of one graph in $\mathcal{G}(S)$ to produce another one in $\mathcal{G}(S)$. This produces the so called flip graph, whose vertices are the elements of $\mathcal{G}(S)$ and whose edges are the flips among them. Three central questions are: (1) Is the flip graph connected? (2) What is the diameter (or radius) of the flip graph? (3) What is the complexity of finding the shortest flip sequence between two given elements?

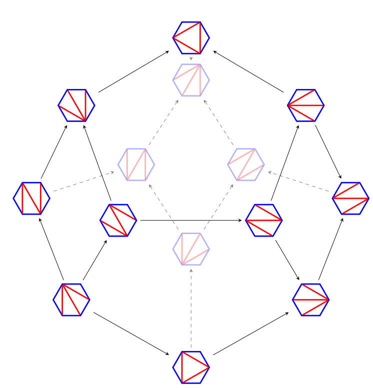


Figure 1: The flip graph of triangulations of a hexagon.

These questions have inspired a substantial amount of research on various reconfiguration problems, some of which have been solved for special families of graphs, and some of which are still open. In this project, we concentrate on specific related open problems for some families of graphs on S: plane spanning paths, plane spanning trees, triangulations, and pointed pseudo triangulations.

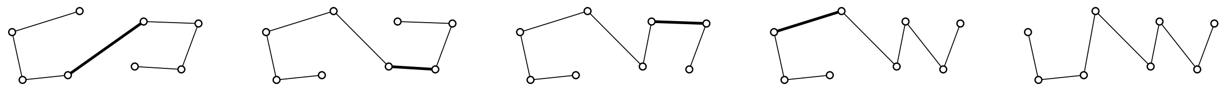


Figure 2: Flipping to an x-monotone path in four steps. In every step the bold edge is flipped.

For a concrete example consider the set $\mathcal{P}(S)$ of all plane spanning paths for a given point set S. Let P_1 and P_2 be two paths from $\mathcal{P}(S)$. Can you transform P_1 into P_2 by a sequence of flip operations, that is, by exchanged single edges step by step, c.f. Figure 2? It it important to observe that all resulting intermediate graphs have to be plane spanning paths from $\mathcal{P}(S)$.

A famous problem is the reconfiguration of triangulations of a point set in the plane, c.f. Figure 1. Here, a flip replaces a diagonal edge in a convex quadrilateral of the triangulation by the other diagonal. For points in convex position, this is (via its dual) equivalent to the rotation of an edge in a given (abstract) rooted binary tree. This is of interest in the study of splay trees, a central data structure for algorithms. While (tight) upper bounds for the reconfiguration distance are well known the complexity of computing the shortest distance between two triangulations of a convex point set (and thus between two binary splay trees) is still an open problem.

We also aim to provide better lower and upper bounds for the number of elements of $\mathcal{G}(S)$ among all point configurations S with n points, in the cases where no tight bounds are known so far.

Supervising faculty members Oswin Aichholzer (http://www.ist.tugraz.at/aichholzer/) and Cesar Ceballos (http://www.geometrie.tugraz.at/ceballos/). If you have any further questions, feel free to contact them.

Oswin Aichholzer is vice head of the Institute of Software Technology at TU Graz. He is a recognized expert in discrete and computational geometry, graph algorithms, and enumerative combinatorics. Aichholzer has supervised 10 PhD students (3 currently under supervision and 7 graduated). Three of his former students have a tenured position in academia. He is editor of the Journal of Computational Geometry and was head of two large research projects (a National Research Network and a EUROCORES multinational Collaborative Research Project).

Cesar Ceballos is an assistant professor (non tenure-track with habilitation) at the Institute of Geometry at TU Graz, working on discrete geometry and algebraic combinatorics. He is the head of an FWF Stand-Alone Project and the Austrian coordinator of a large ANR-FWF International Collaborative Research Project between Austria and France. Among his distinctions, he received a *Banting Award*, the most prestigious award for postdoctoral researchers of the Government of Canada.