Topic 10: Combinatorial optimization on graphs and their drawings

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Scientific background. Discrete geometry and combinatorial optimization have a rich interplay. Many optimization problems are NP-hard for general inputs, but become efficiently solvable/approximable for restricted yet important classes of inputs, for example, for certain graph and matrix classes or when geometric structures play a role. Graphs and their drawings are a central object of study in mathematics and computer science as well as in this project. We consider drawings of graphs where the vertices are represented as points in the plane and the edges are represented by simple curves (or line segments, in straight-line drawings) connecting the points. In simple drawings, any two curves intersect in at most one common point.

In the context of optimization problems on graphs and their drawings, complete graphs constitute an especially interesting and challenging object of study: For example, the crossing number problem (how many crossings any drawing of a graph has at least) is NP-hard for general graphs [4]. But the special case of complete graphs is unlikely to be computationally hard (confer the famous Harary-Hill conjecture [1, 6]). Similarly, the c-colored crossing number problem (find the smallest k such that the edges of a given graph drawing can be c-colored in a way that the number of monochromatic crossings is at most k) is NP-hard for general graphs already for $c = 2$ [8], while its complexity status for drawings of complete graphs is open. One of the few known hardness results for complete graphs is whether a given simple drawing of the complete graph K_n contains a plane subdrawing with $\geq k$ edges [3]. The corresponding problem for straight-line drawings of K_n is easy as there every maximal plane subdrawing is a triangulation and is also maximum.

The study of simple drawings and questions on them is closely related to intersection graphs, as every (simple) drawing D of a graph induces an intersection graph. Hence, identifying structural properties of such graphs is a promising step towards improved optimization algorithms.

Aims. The objective of this project is to study combinatorial optimization problems on graphs which are derived from an underlying geometric context. Every (simple) drawing D of a graph induces an intersection graph, which we refer to as its crossing graph $\mathrm{Cr}(D)$. In $\mathrm{Cr}(D)$ the nodes correspond to the edges (curves) of the drawn graph and two nodes are adjacent if the corresponding edges cross in the drawing. Many optimization problems on drawings of graphs have equivalent formulations on their crossing graphs, including the above-mentioned ones. For example, the c-colored crossing number of a drawing D corresponds to a maximum c-cut in $\mathrm{Cr}(D)$ and a maximum plane subdrawing of D corresponds to a maximum independent set in $\mathrm{Cr}(D)$. The Maximum-Cut and the Maximum-Independent-Set problem are two examples of classical NP-hard combinatorial optimization problems for which also various classes of efficiently solvable special cases have been studied (see e.g. [2, 5, 9, 7] and references therein). However, the question of what key properties distinguish crossing graphs (in particular, of complete graphs) from general intersection graphs and as such could influence the complexity of computational problems on them is wide open.

Approach and methods. The main objects of study in the course of this project are crossing graphs of graph drawings, other types of intersection graphs, and sets of points in the plane, with a focus on combinatorial problems on those objects. This study will especially include structural properties, feasibility results, complexity questions, and insights concerning the borderline between easy and hard cases for the considered problems, as well as (exact and approximative) algorithms.

One key focus of our investigations are optimization problems on straight-line and simple drawings of graphs. Interesting questions in this context range from different types of crossing numbers of graphs and their drawings to the size and number of plane substructures in different classes of drawings of graphs. Closely related, the second focus are (optimization problems on) various types of intersection graphs and possibly other types graphs derived from an underlying geometric structure. In particular, we aim at determining key properties that distinguish crossing graphs (of different classes of drawings of complete graphs) from general segment intersection graphs or 1-string graphs. These will in turn foster new algorithmic as well as coarse-grained and fine-grained complexity results and yield insights about the borderline between easy and hard cases of NP-hard optimization problems on graphs which are derived from an underlying geometric structure. Example problems include the above-mentioned Maximum-Cut and Maximum-Independent-Set problem, the Maximum-Clique problem, or the Dominating Set problem.

From a methodological point of view, this project involves approaches from different areas of expertise, especially combinatorial optimization, discrete geometry, algorithmic graph theory, and complexity theory. Depending on the student's background and interests, the project offers a wide spectrum of research questions, ranging from low-risk starting problems to longstanding open questions and from combinatorial/structural to algorithmic ones.

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