Bootstrap percolation in high-dimensional product graphs

Scientific Background Bootstrap percolation [5] is a particular type of cellular automata defined on graphs, which has been used to study a wide variety of physical processes such as crystal field interactions, the emergence of magnetic properties, the spread of infections and more [9]. In bootstrap percolation, an initial set of *infected* vertices spread through the graph, with a vertex becoming infected once its number of infected neighbours passes a certain threshold $r \in \mathbb{N}$.

As with many areas of combinatorics, bootstrap percolation has been broadly studied from two differing perspective, the *extremal* and the *probabilistic*. From the former perspective, we are interested in the extremal properties of this model - what is the size and structure of minimal *percolating sets* [12], i.e, initial sets such that eventually all vertices become infected, and how is this related to the geometry of the underlying graph. From the latter perspective, we are interested in the average case behaviour of this model - for a *random* initial set, what is the probability that it percolates, and how does this probability evolve as we increase the density of the initial set? In particular, we are interested in the location of the *percolation threshold*, the probability at which it becomes more likely than not that a random initial set will percolate and the size of the *critical window*, the range of infection probabilities where the probability of percolation transitions from almost never to almost surely.

Such questions have been well-studied in a variety of random graph models [13, 15], where they have led to the development of many important techniques in the study of percolation models on graphs, and also in many highly structured, lattice-like graphs [2, 4], such as high-dimensional grids and tori, which arise naturally when considering the physical processes motivating the model.

Hypotheses/Aims One particularly interesting class of graphs on which bootstrap percolation has been studied are hypercubes. The hypercube is a fundamental object of study in diverse areas of maths and computer science, and represents perhaps the simplest example of a high-dimensional model with non-trivial geometry on which to study percolation. In particular, recently the percolation threshold was determined in the hypercube Q^d under two particular simple update rules, 2-neighbour percolation [1] and majority percolation [3]. We propose to study the percolation threshold in this model under more general update rules.

We also propose to study generalisations of these questions in *high-dimensional product graphs*, graphs arising as the Cartesian product of many graphs. This family of graphs includes many well-studied examples of high-dimensional graphs such as hypercubes, grids, tori and Hamming graphs. Recently, a more systematic study of percolation processes on such graphs has been initiated [11], and we propose to extend this to bootstrap percolation.

In all cases it would also be interesting to determine the size of the critical windows, and how the process stabilises in this regime.

Approaches/Methods In order to study these questions we will use combinatorial and discrete probabilistic methods that have proven useful for studying other percolation problems. In particular, it will be useful to enumerate minimal percolating sets in these graphs, which may require analytic techniques such as the use of generating functions, and to determine their structure. To this end it will be useful to analyse the r-core of the uninfected sites, which may be possible by analysing a *pruning process* which can be done for example using branching processes [17], the differential equation method [16], or the general model of Warning propagation [7, 8]. The isoperimetric properties of the hypercube are well-known [14], and have proven to be key tools in the investigation of percolation models on Q^d. More recently the isoperimetric properties of high-dimensional product graphs [6, 10] have been studied, and they will be key to understanding bootstrap percolation in these graphs.

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Mihyun Kang is a full professor at TU Graz and leads the Combinatorics Group. Her main research areas are combinatorics, discrete probability, and algorithms. She is best known for her work on topological properties of random graphs. She received a prestigious *Friedrich Wilhelm Bessel Research Award* of the Alexander von Humboldt Foundation. She serves on the editorial board of leading journals, including *Random Structures and Algorithms*. She supervised five PhD students and currently supervises two postdocs and two PhD students.

Joshua Erde is an assistant professor (non-tenure track with habilitation) at the Institute of Discrete Mathematics at TU Graz. His main research focuses are in random structures, isoperimetric problems, and structural and infinite graph theory. He received a competitive research fellowship of the *Alexander von Humboldt Foundation* and is currently the PI the FWF stand-alone project "Supercritical behaviour in random subgraph models". He currently supervises a postdoc and co-supervises two PhD students.

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