ARITHMETIC AND IDEAL THEORY OF COMMUTATIVE DOMAINS VIA ADDITIVE COMBINATORICS

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Ideal and Factorization Theory of Commutative Rings. Ideal theory of commutative rings originated in the 19th century with Dedekind's work on algebraic numbers. In the first half of the 20th century, it became an independent research subject related to both algebraic number theory and algebraic geometry, and was developed by Krull, Noether, and Prüfer. Multiplicative ideal theory (advanced by Gilmer in the 1970s) studies the multiplicative structure of rings and their associated semigroups of (nonzero, invertible, divisorial, etc.) ideals.

Large parts of the theory can be derived in a purely multiplicative setting without reference to the additive structure of the underlying rings. The first axiomatic foundation of an ideal theory for monoids is due to Lorenzen, and this theory was further developed by Jaffard, Aubert, and Halter-Koch.

Factorization theory describes the arithmetic structure of rings and their semigroups of ideals by arithmetic invariants, such as sets of lengths. As in the case of multiplicative ideal theory, large parts of factorization theory can be developed in the abstract setting of commutative semigroups. This abstract approach allows the development of a powerful transfer machinery, bringing factorization theory within the reach of discrete methods.

This strategy has proved successful in the case of Krull monoids and domains, but only first steps have been taken for other classes of rings. In the Krull domain case we have a strong understanding of divisorial ideals, making it possible to construct a transfer homomorphism from a Krull domain to an associated monoid of zero-sum sequences over its class group. Zero-sum sequences over Abelian groups are then studied with methods from additive combinatorics, and the arithmetic results pulled back to the Krull domain via the transfer homomorphism.

Current Methods in Ideal and Factorization Theory. Let D be a commutative integral domain. If D satisfies the ascending chain condition on principal ideals then every nonzero nonunit $a \in D$ has a factorization $a = u_1 \ldots u_k$, with $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in D$ irreducible. The number k is called the length of the factorization and the set $\mathsf{L}(a)$ of all lengths of factorizations of a is called the set of lengths of a. The domain D is called factorial if the factorization of any given element into irreducibles is essentially unique. This is equivalent to D being a Krull domain with trivial class group.

The domain D is a Krull domain if and only if its monoid of nonzero elements is a Krull monoid, or, equivalently, if its monoid of invertible ideals is a Krull monoid. Every Krull monoid H allows a transfer homomorphism θ to a monoid of zero-sum sequences, say $\theta \colon H \to \mathcal{B}(G_0)$, where G_0 is a subset of the class group of H. Zero-sum sequences are studied with methods from additive combinatorics (e.g., [8, 9]) and transfer homomorphisms allow us to pull back arithmetic results from the combinatorial object $\mathcal{B}(G_0)$ to the Krull monoid H.

So far all this is classic. In recent years, deep results have revealed that some non-Krull monoids allow transfer homomorphisms to monoids of zero-sum sequences [1, 2, 12, 13]. These non-Krull monoids include large classes of non-commutative rings, of non-cancellative monoids of modules, and of commutative Noetherian domains that are not integrally closed [7]. On the other hand, we have constructed examples of Prüfer domains that do not allow transfer homomorphisms to any monoid of zero-sum sequences [4, 6]. (Intriguingly, in these examples the multiplicative monoids consisting of all factors of a fixed element are Krull monoids [5]).

Open Problems and Research Topics. We briefly describe three aims of the present project. First, extrapolating from the known examples and counterexamples, we seek a characterization of those monoids and domains that allow a transfer homomorphism to a monoid of zero-sum sequences (in technical terms, transfer Krull monoids and domains; see [3]). Our focus will be on Prüfer BF-domains.

Second, for transfer Krull monoids – for which the existence of a transfer homomorphism is known on theoretical grounds – we seek an explicit transfer homomorphism. This involves determining the class group and, what is more, the distribution of height-one prime ideals in the classes. Our focus will be on the monoid of invertible ideals of a Krull domain.

Third, once an explicit transfer homomorphism is known, we seek to apply methods of additive combinatorics concerning sets of lengths of zero-sum sequences over finite (or, finitely generated) Abelian groups to derive information about the arithmetic of the monoid or domain. About these methods, see the survey by Schmid [11] and the recent book by Grynkiewicz [10].

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