Information Theory SS 2025

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Exercise sheet 2 (27.03.2025)

Exercise 2.1. Let X be a binomially distributed random variable with parameters $n \in \mathbb{N}$ and $q \in [0,1]$, i.e. $X \sim Bin(n,q)$. Show:

$$\mathbb{E}\left(\frac{1}{1+X}\right) = \frac{1-(1-q)^{n+1}}{(n+1)q}\,.$$

Exercise 2.2. Let (X,Y) be a discrete two-dimensional random variable, where X takes values from $\{-2,-1,0\}$ and Y from $\{-1,1\}$. The following probabilities are known:

$$\mathbb{P}[X=-2, Y=-1] = 0.1, \qquad \mathbb{P}[X=-1] = 0.3,$$
$$\mathbb{P}[X=-2, Y=1] = 0.2, \qquad \mathbb{P}[Y=-1] = 0.4,$$
$$\mathbb{P}[X=-1, Y=1] = 0.3.$$

(a) Calculate the complete table of probabilities of the random vector (X,Y), including the marginal distributions of X and Y.

(b) Calculate the expectation and the variance of X and Y.

(c) Are X and Y independent?

(d) Calculate the covariance $\mathrm{Cov}(X,Y)$ of X and Y, which is defined by

$$\mathrm{Cov}(X,Y) = \mathbb{E}\left((X-E(X))(Y-\mathbb{E}(Y))\right).$$

Exercise 2.3. Let $(X_n)_{n\in\mathbb{N}}$ be an infinite sequence of independent random variables with $\mathbb{P}[X_n = 1] = p_n$ and $\mathbb{P}[X_n = 0] = 1 - p_n$. Show:

(a) $X_n \to 0$ in probability if and only if $p_n \to 0$.

(b) $X_n \to 0$ almost surely if and only if $\sum_{n\geq 1} p_n < \infty$.

Remark: For the second part, you may use the Borel-Cantelli Lemma *without proof, although it can be shown directly.*

Exercise 2.4. Show that a random variable $X : \Omega \to \mathbb{R}$ is almost surely constant, if and only if

$$\mathbb{V}(X) = 0\,.$$

Remark: A random variable X is called almost surely constant*, if* $\mathbb{P}[X = a] = 1$ *holds for some $a \in \mathbb{R}$.*

Exercise 2.5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables and let N be an integer-valued random variable independent of the X_i. Let

$$S = X_1 + \cdots + X_N = \sum_{i=1}^{\infty} X_i \mathbb{1}_{i \leq N}.$$

Show that

$$\mathbb{V}(S) = \mathbb{V}(X_1)\mathbb{E}(N) + (\mathbb{E}(X_1))^2 \, \mathbb{V}(N).$$

Exercise 2.6. Given a non-negative integer-valued random variable X, show that

(a) $\displaystyle\sum_{i=0}^{\infty} i \, \mathbb{P}[X \geq i] = \frac{\mathbb{E}(X^2) + \mathbb{E}(X)}{2}$,

(b) $\displaystyle\sum_{i=0}^{\infty} i \, (\mathbb{P}[X \geq i] + \mathbb{P}[X > i]) = \mathbb{E}(X^2)$.