Information Theory SS 2025

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Exercise sheet 3 (10.04.2025)

Exercise 3.1. Let X and Y be the values of two independently thrown six-sided dice. Calculate

(a) $H(X) + H(Y)$,

(b) $H(X + Y)$,

(c) $H(X - Y)$,

(d) $H(\max(X, Y))$.

Exercise 3.2. Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subseteq \mathbb{R}$. Let $x_1, \dots, x_n \in I$ be points in the interval and let $p_i \in [0, 1]$ for $i = 1, \dots, n$ be probabilities, i.e., $p_1 + \dots + p_n = 1$. Show that:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i).$$

Exercise 3.3. A fair coin is flipped until heads occurs the first time. Let X denote the number of flips required.

(a) Find the entropy $H(X)$ in bits.

(b) Find an efficient sequence of yes-no questions in order to guess the outcome of X. Compare $H(X)$ to the expected number of questions.

Exercise 3.4. The NBA Playoffs are a best-of-seven series between Team A and Team B, which ends when one team wins four games. Let the random variable X represent the outcome of the series of games. For example, some possible outcomes are AAAA, ABABABA, and AAABBBB. Let the random variable Y represent the number of games played. In the previous three examples this is 4, 7, and 7, respectively. Assume that the two teams are equally strong. Find the values of $H(X)$, $H(Y)$, $H(Y \mid X)$, and $H(X \mid Y)$.

Exercise 3.5. Let X and Y be random variables that take values in finite sets $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $Z = X + Y$.

(a) Show that $H(Z \mid X) = H(Y \mid X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$.

(b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

(c) Find a necessary and sufficient condition for $H(Z) = H(X) + H(Y)$.

Exercise 3.6. An urn contains r red, w white, and b black balls. We first draw $k \geq 2$ balls (one by one) from the urn with replacement (putting them back after each draw). Denote their colours by $X_1, \dots, X_k$. Then we draw k balls from the urn without replacement. Denote their colours by $Y_1, \dots, Y_k$.

(a) Find $H(X_1, \dots, X_k)$.

(b) Which is larger, $H(X_1, \dots, X_k)$ or $H(Y_1, \dots, Y_k)$? Prove your statement.

(*Hint: Clearly Y_1 and X_1 have the same distribution. Show that also Y_k and X_1 have the same distribution for all k.*)

Exercise 3.7. Suppose you have $n > 2$ coins and among them there may or may not be a counterfeit coin, which is either heavier or lighter than the other coins. You weigh the coins with a balance to determine if there is a heavier or lighter counterfeit coin. To be precise, in a weighing you take two disjoint sets A and B of coins and determine which set is heavier than the other, or whether they weigh the same. Let $k(n)$ be the smallest number of weighings always sufficient to find the counterfeit coin (if any) among the n coins and correctly declare it to be heavier or lighter.

(a) Show that $k(n) \geq \log_3(2n + 1)$.

(b) (Difficult) Give a coin weighing strategy for $k = 3$ weighings and $n = 12$ coins.

Exercise 3.8. Show that the entropy of the probability distribution

$$(p_1, \dots, p_i, \dots, p_j, \dots, p_n)$$

cannot be larger than the entropy of the distribution

$$(p_1, \dots, \frac{p_i + p_j}{2}, \dots, \frac{p_i + p_j}{2}, \dots, p_n).$$

When do we have equality?