Information Theory SS 2025

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Exercise sheet 5 (22.05.2025)

Exercise 5.1. Consider the *Drunkard's walk* Markov chain with state space $\mathcal{X} = \{0, 1, \ldots, N\}$ and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \beta & 0 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 0 & \alpha & 0 \\ 0 & 0 & \cdots & 0 & \beta & 0 & \alpha \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $0 < \alpha < 1$ is the probability of moving one step from position k to position $k+1$, and $\beta = 1 - \alpha$ is the probability to move from position k to position $k-1$, for $k = 1, \ldots, N-1$.

(a) Given an initial distribution $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the j-th entry, let p_j, for $j = 0, \ldots, N$, be the probability that $X_n = N$ for some $n \geq 0$ (the drunkard reaches home). Find a set of linear equations for the p_j. [Hint: Express p_j in terms of p_{j-1} and p_{j+1}]

(b) Compute p_j for the concrete case where $N = 3$, $j = 1$ and $\alpha = \frac{1}{2}$.

Exercise 5.2. Consider the *Weather in Oz* Markov chain $(X_n)_{n\geq 0}$ with state space $\mathcal{X} = \{c, r, s\}$ (clear, rainy, snowy) and transition matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

(a) Compute a stationary distribution for the Markov chain.

(b) Compute the entropy rate of the stochastic process.

Exercise 5.3. Consider the experiment of consecutively rolling a fair die. For $n \geq 1$, let Z_n be the RV representing the number of rolls until '6' appears n times.

(a) Is $(Z_n)_{n\geq 1}$ a Markov Chain?

(b) Is $(Z_n)_{n\geq 1}$ stationary?

(c) Compute the entropy rate of $(Z_n)_{n\geq 1}$.

Exercise 5.4.

(a) Prove that for every Markov chain $(X_n)_{n\in\mathbb{N}}$ the quantity $H(X_0|X_n)$ is non-decreasing in $n \geq 0$.

(b) Show that for any stationary discrete stochastic process $(X_n)_{n\in\mathbb{Z}}$,

$$H(X_0|X_{-1}, \dots, X_{-n}) = H(X_0|X_1, \dots, X_n).$$

Exercise 5.5. We are given 2 coins: when tossed, coin A shows 'head' with probability p and 'tail' with probability $1-p$, coin B is a fair coin. Consider the following two-stage experiment. At the first stage we choose coin A with probability q and coin B with probability $1-q$. At the second stage we toss repeatedly and independently the chosen coin. Let X_0 be the RV taking value 1 if coin A is chosen and the value 0 if coin B is chosen. For $n \geq 1$, let X_n be the RV referring to the outcome of the n-th toss: if 'head' shows up then $X_n = 1$ and if 'tail' shows up then $X_n = 0$.

(a) For which values of p and q is $(X_n)_{n\geq 1}$ stationary?

(b) For which values of p and q is $(X_n)_{n\geq 0}$ stationary?

(c) Compute the entropy rates of $(X_n)_{n\geq 1}$ and $(X_n)_{n\geq 0}$.

Exercise 5.6. Let $(X_n)_{n\geq 1}$ be a time-homogeneous Markov chain with state space $\mathcal{X} = \{0, 1\}$ and transition probabilities: $p(0|0) = 0.3$, $p(1|0) = 0.7$, $p(0|1) = 0.2$, $p(1|1) = 0.8$.

(a) Draw the transition graph.

(b) Compute the stationary distribution ν of the Markov chain.

(c) Let τ^0 be the return time to 0 after starting with $X_0 = 0$. Based on the transition graph, compute $\mathbb{E}(\tau^0 \mid X_0 = 0)$ as the sum of an infinite series and then find $\nu(0)$ according to the Ergodic Theorem for Markov Chains. Compare it with what you computed in (b).

Exercise 5.7. Imagine you are walking on the integers in the direction you are facing (left or right), reversing direction after each step taken with probability $p = 0.2$. You start at 0 facing to the right. For $n \geq 0$ let X_n be your position at time n.

(a) Describe the state space and transition probabilities of this process. Is $(X_n)_{n\geq 0}$ a Markov chain?

(b) What is your expected number of steps taken before reversing direction?

(c) Calculate $H(X_1, \dots, X_n)$.

(d) Find the entropy rate of this process.

Exercise 5.8. Prove the following generalization of the claim used in the proof of Proposition 3.19:

Let $(X_n)_{n\geq 0}$ be an irreducible time-homogeneous Markov chain with a finite state space $\mathcal{X}$ and transition matrix P. If $f : \mathcal{X} \to \mathbb{R}$ is a harmonic function with respect to P for all $x \in \mathcal{X}$ except for at most one point $x_0 \in \mathcal{X}$, that is, when viewed as a column vector $(Pf)_x = f(x)$ for all $x \in \mathcal{X} \setminus \{x_0\}$, then $f = c\mathbf{1}$ is constant.