Information Theory SS 2025

Institut für Diskrete Mathematik (5050), TU Graz

Exercise sheet 7 (26.06.2025)

Exercise 7.1. A channel has the following probability transition matrix:

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Show that its capacity is achieved by a distribution that places zero probability on one of the input symbols. Calculate the capacity of the channel. Give an intuitive reason why that letter is not used.

Exercise 7.2. Consider two independent discrete channels $\mathcal{C}_1 = (\mathcal{X}_1, P_1, \mathcal{Y}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, P_2, \mathcal{Y}_2)$. We construct a new channel

$$\mathcal{C} = (\mathcal{X}_1 \times \mathcal{X}_2,\ P,\ \mathcal{Y}_1 \times \mathcal{Y}_2)$$

from $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ will be transmitted in parallel: for $i \in \{1, 2\}$ the input $x_i \in \mathcal{X}_i$ is sent to some $y_i \in \mathcal{Y}_i$ according to the transition matrix P_i. Show that the channel capacity satisfies $\operatorname{cap}(\mathcal{C}) = \operatorname{cap}(\mathcal{C}_1) + \operatorname{cap}(\mathcal{C}_2)$.

Exercise 7.3. Given a channel $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ suppose we form a new channel $\mathcal{C}'$ by transmitting the same message X across the channel twice, resulting in an output (Y_1, Y_2) where Y_1 and Y_2 are conditionally independent and conditionally identically distributed given X.

(a) Show that
$$I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2).$$

(b) Conclude that $\operatorname{cap}(\mathcal{C}') \leq 2 \operatorname{cap}(\mathcal{C})$.

Exercise 7.4. Let $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ be a binary channel which is composed of two binary channels in sequence, such that the output of the first channel $\mathcal{C}_1 = (\mathcal{X}, P_1, \mathcal{Z})$ is the input of the second channel $\mathcal{C}_2 = (\mathcal{Z}, P_2, \mathcal{Y})$. Let the transition matrices be

$$P_1 = \begin{bmatrix} 1 - p_1 & p_1 \\ p_1 & 1 - p_1 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 - p_2 & p_2 \\ p_2 & 1 - p_2 \end{bmatrix}.$$

(a) Show that $\operatorname{cap}(\mathcal{C}) \leq \min\{\operatorname{cap}(\mathcal{C}_1), \operatorname{cap}(\mathcal{C}_2)\}$.

(b) Compute the transition matrix P and the channel capacity $\operatorname{cap}(\mathcal{C})$.

(c) Assume that $0 \leq p_1 \leq p_2 < 1/2$. When do we have equality in (a)?

Exercise 7.5. Let $\mathcal{C} = (\mathcal{X}, P, \mathcal{Y})$ be a discrete memory-less channel with capacity C. Suppose that this channel is immediately followed by an erasure channel, that erases the output of $\mathcal{C}$ with probability α. This yields a new channel $\mathcal{C}' = (\mathcal{X}, P', \mathcal{Y} \cup \{e\})$ with transition probability

$$p'(y \mid x) = (1-\alpha)p(y \mid x) \text{ and } p'(e \mid x) = \alpha \text{ for } x \in \mathcal{X}, y \in \mathcal{Y}.$$

Compute the capacity of $\mathcal{C}'$.

Exercise 7.6. Consider a channel with binary inputs that has both errors (with probability ϵ) and erasures (with probability α), and so has transition matrix

$$P = \begin{pmatrix} 1-\alpha-\epsilon & \epsilon & \alpha \\ \epsilon & 1-\alpha-\epsilon & \alpha \end{pmatrix}.$$

Find the capacity of this channel.