Horn's problem, projection of orbital measures, and multivariate spline functions

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Assume that the eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ of two Hermitian matrices A and B are known. What can be said about the eigenvalues of the sum $A + B$? This is Horn's problem. In 1962 Horn proposed a conjecture in terms of a system of inequalities. This conjecture has been proven by Klyachko in 1998. Following Frumkin and Goldberger (2006), and Coquereaux and Zuber (2018), we consider Horn's problem from a probabilistic viewpoint. The set $\mathcal{O}_\alpha$ of the Hermitian matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ is an orbit for the action of the unitary group $U(n)$ acting on the space $\mathcal{H}_n$ of Hermitian matrices. Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$ and, independently, the random Hermitian matrix Y on the orbit $\mathcal{O}_\beta$. The question is now: What is the joint distribution of the eigenvalues of the sum $X + Y$? We have established a formula for this distribution. It involves projections of orbital measures on the subspace D_n of diagonal matrices. The density of this distribution is piecewise polynomial, a multivariate spline function.

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