The 50 Fake Projective Planes

Donald CARTWRIGHT
University of Sydney

Joint work with

Tim STEGER
University of Sassari

A fake projective plane is a smooth compact complex surface P which is not biholomorphic to the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, but has the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$, namely 1, 0, 1, 0, 1.

The first fake projective plane was constructed by Mumford in 1979. He also showed that there could only be a finite number of these surfaces.

Two more examples were found by Ishida and Kato in 1998, and another by Keum in 2006.

In their 2007 Inventiones paper [PY], Gopal Prasad and Sai-Kee Yeung almost completely classified fake projective planes. They showed that these fall into a small number of "classes".

For 28 of the classes, they give at least one fake projective plane. These 28 classes are all defined using unitary groups associated with certain **cubic division algebras**.

For a very small number of classes, they left open the question of existence of fake projective planes in that class, but conjectured that there are none. These classes are all defined using certain unitary **matrix groups**.

All the classes are associated with pairs (k, ℓ) of fields, and extra data. It turns out that k is either $\mathbb{Q}$ or a real quadratic extension of $\mathbb{Q}$, and ℓ is always a totally complex quadratic extension of k.

Here is an example of one of the cubic division algebras which arises:

The fields involved are $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-7})$.

The algebra is exhibited as a "cyclic simple algebra" by using an auxiliary field: $m = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_7$.

This is a degree 3 extension of ℓ (note that $\sqrt{-7} = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4$) with Galois group $\mathrm{Gal}(m/\ell) = \langle \varphi \rangle$, where $\varphi(\zeta) = \zeta^2$.

The algebra is

$$\mathcal{D} = \Big\{ a + b\sigma + c\sigma^2 \ : \begin{array}{l} \bullet\ a, b, c \in m, \\ \bullet\ \sigma x \sigma^{-1} = \varphi(x) \text{ for all } x \in m, \\ \bullet\ \sigma^3 = \dfrac{3+\sqrt{-7}}{4}. \Big\} \end{array}$$

What Tim and I have been doing is

(a) finding **all** the fake projective planes, up to isomorphism, in each of the 28 division algebra classes, and

(b) showing that there are indeed no fake projective planes in the unresolved matrix algebra classes.

In (a), we find that there are, altogether, **50** fake projective planes.

This count depends on what "isomorphism" means. If it means "biholomorphism", then we should multiply this number by 2. We are in fact classifying the fpp's according to their **fundamental groups**. It follows from a result of Siu, that if two fpp's have isomorphic fundamental groups, then they are either biholomorphic or conjugate-biholomorphic.

Recall that $U(2,1)$ is the group of 3×3 complex matrices g such that $g^* F_0 g = F_0$, where

$$F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and that $PU(2,1)$ is $U(2,1)$, modulo scalars. This is the automorphism group of $\mathbb{B}^2_{\mathbb{C}} = \{(z_1, z_2) \ : \ |z_1|^2 + |z_2|^2 < 1\}$, with the hyperbolic metric.

Theorem (Klingler, Yeung). The fundamental group Π of a fake projective plane is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$.

So a fake projective plane is a ball quotient $\mathbb{B}^2_{\mathbb{C}}/\Pi$ for such a Π.

In [PY], for each of the classes they define a cocompact lattice subgroup $\bar{\Gamma}$ of $PU(2,1)$. They calculate the covolume $\mu(PU(2,1)/\bar{\Gamma})$. It must be of the form $1/N$ for an integer N.

For the 28 cubic division algebra cases, the only possibilities for N are 1, 3, 9 and 21. In the matrix algebra cases, N is one of 48, 288, 600, and 864.

Theorem [PY]. The fundamental groups of the fake projective planes in this class are the subgroups Π of $\bar{\Gamma}$ such that

(a) $[\bar{\Gamma} : \Pi] = N$,

(b) Π is torsion-free, and

(c) $\Pi/[\Pi, \Pi]$ is finite.

Our main result: For each of the $\bar{\Gamma}$'s we have found **generators** (explicit elements of the corresponding cubic division algebra or matrix algebra). We have found a **presentation** of $\bar{\Gamma}$ in each case.

This allows us to find all subgroups Π in $\bar{\Gamma}$ (up to conjugacy) satisfying (a), (b) and (c). The computer algebra package Magma, for example, has a command `LowIndexSubgroups(G,n)` which lists all conjugacy classes of subgroups of a given index n in a finitely presented group G.

Magma's command `Rewrite(G,H)` finds a presentation of a given subgroup H of finite index in a given finitely presented group G.

So the fundamental groups Π of the fpp's can be listed, and presentations given for each of them.

Example. For the class we call $(a = 7, p = 2, \{7\})$, $\bar{\Gamma}$ has the presentation

$$\begin{aligned}\bar{\Gamma} = \langle\ b, z\ |\\ &b^3 = 1,\\ &z^7 = 1,\\ &(bz^{-2}bz^{-1})^3 = 1,\\ &z^2b^{-1}z^{-1}bz^2b^{-1}zb^{-1}zbz^2b = 1,\\ &b^{-1}z^{-2}bz^2b^{-1}zb^{-1}z^{-1}bz^2b^{-1}z^{-1}b^{-1}z^{-2} = 1,\\ &z^{-1}bz^2b^{-1}z^{-1}bz^{-1}bz^2b^{-1}z^{-1}bz^{-1}bz^{-2} = 1,\\ &bz^{-2}b^{-1}z^3bz^{-2}b^{-1}z^{-1}bz^{-2}b^{-1}z^{-2}b^{-1}z^3bz^2b^{-1}zbz = 1\rangle.\end{aligned}$$

In this case $\mu(PU(2,1)/\bar{\Gamma}) = 1/21$. Magma's command

`LowIndexSubgroups`$(\bar{\Gamma}, 21)$

lists all (conjugacy classes of) subgroups of $\bar{\Gamma}$ of index at most 21.

Example (continued). It finds that there are four subgroups of index 21:

$$\Pi_a = \langle z^3b^{-1}zb,\ zb^{-1}zbz^2,\ bz^2b^{-1}z^3\rangle,$$

$$\Pi_b = \langle zb^{-1}zbz,\ z^{-1}b^{-1}zb^{-1}zb^{-1},\ bz^{-1}bz^2b\rangle,$$

$$\Pi_c = \langle [b^{-1}, z^{-1}],\ z^2bz^2b^{-1},\ z^2b^{-1}z^2b\rangle,$$

$$\Pi_d = \langle bz^{-1},\ b^{-1}z^3,\ (z^2b^{-1}z)^2\rangle.$$

All are torsion-free with finite abelianization.

Our calculations also give us a list of conjugacy classes of elements of finite order in each $\bar{\Gamma}$.

This helps us give some interesting singular surfaces X_G corresponding to groups G such that $\Pi < G \leq \bar{\Gamma}$, because then

- $\pi_1(X_G) \cong G/\langle$ torsion elements in $G \rangle$.

For the above example, $\bar{\Gamma}$ is generated by the elements b and z, which have finite order. So $\pi_1(X_{\bar{\Gamma}})$ is trivial.

Also, the subgroup

$$G = \langle \bar{\Gamma} \mid z,\ bzb^{-1} \rangle$$

has index 3 in $\bar{\Gamma}$. For three of the above Π's, $\Pi < G \leq \bar{\Gamma}$. Also, G is clearly generated by elements of finite order, and so $\pi_1(X_G)$ is trivial.

There are several examples of this sort in our list. Most arise from examples with $[G : \Pi] = 3$, and then the $\pi_1(X_G)$'s (coming from various classes) are:

$$\{1\},\ C_2,\ C_3,\ C_4,\ C_7,\ C_{13},\ C_2 \times C_3,\ C_2 \times C_7,$$
$$C_2 \times C_2,\ C_2 \times C_4,\ S_3,\ D_8,\ Q_8.$$

We also have examples of G's with $\pi_1(X_G)$ trivial and examples with $\pi_1(X_G) = C_2$ for $[G : \Pi] = 7$, 9 and 21.

By knowing a presentation for each Π, we can easily determine whether Π can be lifted to $SU(2,1)$.

In geometric terms, this is equivalent to asking whether the canonical line bundle of $\mathbb{B}^2_{\mathbb{C}}/\Pi$ is divisible by 3.

This was proved in [PY] to be true for most cases, but in the $\mathcal{C}_2$ and $\mathcal{C}_{18}$ cases this issue was left open.

Armed with explicit generators and relations, it is easy to determine whether a given Π ($\subset \bar{\Gamma} \subset PU(2,1)$) lifts to $SU(2,1)$.

In the above example, the generators b and z are initially realized as elements of determinant $(3+\sqrt{-7})/4$ and 1 respectively. Replacing b by tb, where $t^3 = (3-\sqrt{-7})/4$, one checks the relations in the presentation hold in $SU(2,1)$, and gets a lift of all of $\bar{\Gamma}$ to $SU(2,1)$.

In the $\mathcal{C}_{18}$ classes, it turns out that there are Π's which **do not** lift to $SU(2,1)$.

Some of the Π are not congruence subgroups.

An example of a group $\bar{\Gamma}$. This is the $\bar{\Gamma}$ for the class we call $(\mathcal{C}_1, \emptyset)$. It is one of the classes for which we have shown that there is **no** fake projective plane, confirming Gopal and Sai-Kee's conjecture.

The fields k, ℓ involved here are $k = \mathbb{Q}(\sqrt{5})$ and $\ell = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_5$ is $e^{2\pi i/5}$.

$$\begin{aligned} \bar{\Gamma} = \{\xi \in M_{3\times 3}(\ell) \ : \ & \text{(a) } \xi^* F \xi = F, \\ & \text{(b) } \xi \text{ has entries in } \mathbb{Z}[\zeta], \text{ and} \\ & \text{(c) } \det(\xi) = 1.\} \end{aligned}$$

Here

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -g \end{pmatrix},$$

where $g = -\zeta^2 - \zeta^3 = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

In [PY], it is shown that $\mu(PU(2,1)/\bar{\Gamma}) = 1/600$. So to exclude fpp's in this case we proved:

Theorem. $\bar{\Gamma}$ does **not** contain a torsion-free subgroup of index 600.

To show this, we first find matrix generators and a presentation for $\bar{\Gamma}$. To find elements in $\bar{\Gamma}$, and finally a short list of generators, we use the **Cayley transform**.

Write $\iota(\xi) = F^{-1}\xi^* F$. So $\xi^* F \xi = F \Leftrightarrow \iota(\xi)\xi = 1$. Note that

$$\iota \circ \iota = id, \quad \iota(\xi\eta) = \iota(\eta)\iota(\xi) \quad \text{and} \quad \iota(\lambda I) = \bar{\lambda} I.$$

Generalizing the obvious fact that

$$it \text{ purely imaginary} \quad \Rightarrow \quad z = \frac{1+it}{1-it} \text{ satisfies } \bar{z}z = 1,$$

we see that if $M \in M_{3\times 3}(\ell)$, then

$$\iota(M) = -M \quad \Rightarrow \quad \xi = (I+M)(I-M)^{-1} \text{ satisfies } \iota(\xi)\xi = 1.$$

Of course ξ need not be in $M_{3\times 3}(\mathbb{Z}[\zeta])$! We ran C programs to find ξ satisfying this condition and also $\det(\xi) = 1$.

Each entry of M has 4 rational coefficients. The condition $\iota(M) = -M$ is linear in these, and reduces the number of variables from 36 to 18.

Clearing denominators, I in fact worked with $\xi = (dI + M)(dI - M)^{-1}$. So we had 19 integer variables. The determinant condition $\det(\xi) = 1$ gives the degree 3 equation $\det(dI + M) = \det(dI - M)$. Then the arithmetic condition $\xi \in M_{3\times 3}(\mathbb{Z}[\zeta])$ was imposed.

This particular case was quickly dealt with, but in most cases I had to make several choices of d, and let the other variables in M run from -3 to 3. Notice that $7^{18} \approx 1.6 \times 10^{15}$.

Another method we used (in a couple of the cubic division algebra cases) was to choose an element x of the algebra satisfying $\iota(x) = x$ corresponding to a non-compact torus, and to use Magma to find generators of the unit group of $\ell[x]$.

Example (continued). In this case, $\bar{\Gamma}$ is generated by the following four matrices:

$$d_1 = \begin{pmatrix} \zeta^3 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & -\zeta \end{pmatrix}, \quad d_2 = \begin{pmatrix} -\zeta & 0 & 0 \\ 0 & \zeta^3 & 0 \\ 0 & 0 & -\zeta \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -g & g \\ 0 & -1 & g \end{pmatrix}.$$

and has a presentation given by these generators and the relations

$$d_1^{10} = d_2^{10} = w^2 = 1,\ d_1 d_2 = d_2 d_1,\ \text{and } w d_1 w^{-1} = d_2$$

and

$$d_1 a = a d_1,\ a^2 = (a d_1^2 d_2^{-1})^3 = (a d_2^5 w)^5 = (a d_2^6 w a d_2^4 w)^3 = 1.$$

To show that the above matrices generate $\bar{\Gamma}$, we embed $\bar{\Gamma}$ into $SU(2,1)$:

$$\xi \mapsto c\xi c^{-1}, \quad \text{where } c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{g} \end{pmatrix}.$$

As mentioned, [PY] tell us that the covolume of the embedded lattice is $1/600$. So the normalized hyperbolic volume of the fundamental domain

$$\mathcal{F} = \{z \in \mathbb{B}^2_{\mathbb{C}} \ : \ d(0,z) \le d(g(0),z) \text{ for all } g \in \bar{\Gamma}\}$$

is $1/600$. Let Γ' be the subgroup of $\bar{\Gamma}$ generated by d_1, d_2, w and a. Tim has written a program to estimate the hyperbolic volume of

$$\mathcal{F}' = \{z \in \mathbb{B}^2_{\mathbb{C}} \ : \ d(0,z) \le d(g(0),z) \text{ for all } g \in \Gamma'\}$$

I believe that the idea of the program is to consider, for a large set of $z = (z_1, z_2)$ such that $|z_1|^2 + |z_2|^2 = 1$,

$$t_z = \sup\{t \in [0,1) \ : \ d(0,tz) \le d(g(0),tz) \text{ for all } g \in \Gamma' \text{ s.t. } d(g(0),0) \le M\}.$$

The program also estimates (correct to several decimal places) the radius

$$r_0 = \max\{d(z, 0) : z \in \mathcal{F}\}$$

of the fundamental domain. Tim has also proved the following

Theorem. The elements g satisfying $d(g(0), 0) \leq 2r_0$ generate $\bar{\Gamma}$, and the relations of the form $g_1 g_2 g_3 = I$, where $d(g_i(0), 0) \leq 2r_0$ for $i = 1, 2, 3$, are sufficient to give a presentation of $\bar{\Gamma}$.

In the example, there are 68,200 elements $g \in \bar{\Gamma}$ satisfying $d(g(0), 0) \leq 2r_0$. However, they lie in only 5 double-K-cosets $K r_j K$ of the finite subgroup (of order 200)

$$K = \langle d_1, d_2, w \rangle \;\leq\; \bar{\Gamma}.$$

(We can take $r_1 = 1$, $r_2 = a$, $r_3 = awa$, $r_4 = ad_2^2 a$ and $r_5 = ad_2^4 wawa$.) So a relation $g_1 g_2 g_3 = I$ gives us a word in d_1, d_2, w and a involving at most 9 a's.

So the presentation initially involves thousands of relations. This can be given to Magma, which has a command `Simplify`. This and some human intervention led to the above presentation. In this case we could also do this automatically, but without using `Simplify`, by writing specialized string manipulation programs.

Summary of how we showed that $\bar{\Gamma}$ has no torsion-free subgroup of index 600.

The index 600 is well beyond the capabilities of Magma's `LowIndexSubgroups` command.

Suppose that $\bar{\Gamma}$ has a torsion-free subgroup H of index 600. If T is a transversal, then the action of $\bar{\Gamma}$ on T with the property that if $g \in \bar{\Gamma} \setminus \{1\}$ has finite order, then $\boldsymbol{g}$ **acts on** $\boldsymbol{T}$ **without fixed points**:

$$gtH = tH \quad \Rightarrow \quad t^{-1}gt \in H \text{ has finite order.}$$

In particular, this is true for the elements $g \in K \setminus \{1\}$. So we can write $T = Kt_0 \cup Kt_1 \cup Kt_2$, and we know explicitly the action of each $k \in K$ on T.

The action of $\bar{\Gamma}$ on T is known once we know the action of a.

This is a permutation of the 600-element set T with many special properties.

We show, using a back-track style computer search: there is no permutation $a \in \mathsf{Perm}(T)$ with all these properties.

The properties used were:

- ak has finite order for 40 of the 200 elements $k \in K$, $a(t) = k^{-1}t$ cannot hold for any of these k's and any $t \in T$.

- a has order 2 and no fixed points,

- a commutes with the action of d_1,

- $ad_1^2 d_2^{-1}$ and $ad_2^6 w a d_2^4 w$ have order 3 and no fixed points,

- $ad_2^5 w$ has order 5 and no fixed points.

The 28 classes can be divided into groups according to the fields k and ℓ involved. The possible pairs (k, ℓ) are as follows

- For $k = \mathbb{Q}$:
  - $\ell = \mathbb{Q}(\sqrt{-1})$,
  - $\ell = \mathbb{Q}(\sqrt{-2})$,
  - $\ell = \mathbb{Q}(\sqrt{-7})$,
  - $\ell = \mathbb{Q}(\sqrt{-15})$,
  - $\ell = \mathbb{Q}(\sqrt{-23})$,

- $k = \mathbb{Q}\left(\sqrt{5}\right)$, $\ell = k\left(\sqrt{-3}\right)$. "$\mathcal{C}_2$ case".

- $k = \mathbb{Q}\left(\sqrt{2}\right)$, $\ell = k\left(\sqrt{-5} + 2\sqrt{2}\right)$. "$\mathcal{C}_{10}$ case".

- $k = \mathbb{Q}\left(\sqrt{6}\right)$, $\ell = k\left(\sqrt{-3}\right)$. "$\mathcal{C}_{18}$ case".

- $k = \mathbb{Q}\left(\sqrt{7})\right)$, $\ell = k\left(\sqrt{-1}\right)$. "$\mathcal{C}_{20}$ case".

For each of these 9 pairs (k, ℓ), there are at least 2 classes. The division algebra $\mathcal{D}$ involved depends only on (k, ℓ).

Example: the case $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-7})$. There are 6 classes in this case. We use the field $m = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_7$, which is a degree 3 extension of ℓ with Galois group $\mathrm{Gal}(m/\ell) = \langle \varphi \rangle$, where $\varphi(\zeta) = \zeta^2$, and let

$$\mathcal{D} = \Big\{ a + b\sigma + c\sigma^2 \;:\; \begin{array}{l} \bullet\ a, b, c \in m, \\ \bullet\ \sigma x \sigma^{-1} = \varphi(x) \text{ for all } x \in m, \\ \bullet\ \sigma^3 = \dfrac{3+\sqrt{-7}}{4}. \Big\} \end{array}$$

We then define an involution ι on $\mathcal{D}$, and then a group

$$G = \{\xi \in \mathcal{D}^\times \;:\; \iota(\xi)\xi = 1 \text{ and } \mathrm{Nrd}(\xi) = 1\}.$$

so that $G(\mathbb{R}) \cong SU(2,1)$.

There is a simple involution ι_0 on $\mathcal{D}$ which maps σ to σ^{-1} and ζ to ζ^{-1}, and so $\sqrt{-7} = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4$ to $-\sqrt{-7}$.

We replace ι_0 by $\iota : \xi \mapsto w^{-1}\iota_0(\xi)w$, where $w = \zeta + \zeta^{-1}$, to get the desired behaviour $G(\mathbb{R}) \cong SU(2,1)$.

We can embed $\mathcal{D}$ in $M_{3\times 3}(m)$ so that

$$\sigma \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{3+\sqrt{-7}}{4} & 0 & 0 \end{pmatrix}$$

and

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & \varphi(x) & 0 \\ 0 & 0 & \varphi^2(x) \end{pmatrix} \quad \text{for } x \in m.$$

The involution can then be expressed in terms of a diagonal matrix F. For

$$F = \begin{pmatrix} w & 0 & 0 \\ 0 & \varphi(w) & 0 \\ 0 & 0 & \varphi^2(w) \end{pmatrix}$$

we have $\iota(\xi) = F^{-1}\xi^* F$.

In [PY], the possible groups $\bar{\Gamma}$ are found using a group

$$\Lambda = G(\mathbb{Q}) \cap \prod_{p \text{ prime}} P_p,$$

where $(P_p)_{p \text{ prime}}$ is a coherent collection of parahoric subgroups $P_p \leq G(\mathbb{Q}_p)$.

In the current example, in the notation of [PY], the set $\mathcal{R}_\ell$ of primes p which ramify in ℓ is just $\{7\}$. The set $\mathcal{T}_0$ of primes p where G is anisotropic is just $\{2\}$. They show that if $p \neq 2, 7$, P_p is a hyperspecial parahoric subgroup. For $p = 7$, P_p is one of two different "types".

In the current example "$(a = 7, p = 2, \{7\})$", the condition for an element to be in $\bar{\Gamma}$ is that it be the image of an element

$$\xi = \sum_{j=0}^{5} \sum_{k=-1}^{1} c_{jk} \zeta^j \sigma^k,$$

of $\mathcal{D}$ such that

(a) $\iota(\xi)\xi = 1$,

(b) $\mathrm{Nrd}(\xi)$ is a power of $(3 + \sqrt{-7})/4$,

(c) the coefficients c_{jk} are in $\mathbb{Z}[1/2, 1/7]$, and

(d) the vector $\mathbf{c}$ of coefficients satisfies a condition of the form

$$M\mathbf{c} \text{ has entries in } \mathbb{Z}_7 \quad \text{(for a certain } 18 \times 18 \text{ integer matrix } M.)$$

Recall that the group $\bar{\Gamma}$ is generated by two elements b and z. The element z is just the image of ζ_7. The element b is the image of

$$\frac{1}{7}\sum_{j=0}^{5}\sum_{k=-1}^{1} a_{jk}\zeta^j\sigma^k,$$

where the coefficients a_{jk} are the 18 numbers

$$-9, -3, 6, -4, 1, -2, 1, -2, -3, -1, -5, 3, -3, -8, 2, 2, -4, -6$$

in the order

$$a_{0,-1},\ a_{0,0},\ a_{0,1},\ a_{1,-1},\qquad \cdots \qquad,\ a_{5,-1},\ a_{5,0},\ a_{5,1}.$$

In terms of matrices, z and b are the images of

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$$

and $\frac{1}{14}$ times

$$\left(\begin{array}{ccc} -8\zeta^5 - 16\zeta^4 - 10\zeta^3 - 4\zeta^2 + 2\zeta - 6 & -12\zeta^5 + 4\zeta^4 + 6\zeta^3 - 6\zeta^2 - 4\zeta + 12 & 10\zeta^5 + 6\zeta^4 + 2\zeta^3 + 12\zeta^2 + \\ 2\zeta^5 + 4\zeta^4 + 6\zeta^3 - 6\zeta^2 - 4\zeta - 16 & 10\zeta^5 + 6\zeta^4 + 2\zeta^3 + 12\zeta^2 - 6\zeta + 4 & -6\zeta^5 - 12\zeta^4 - 18\zeta^3 - 10\zeta^2 \\ 10\zeta^5 + 13\zeta^4 + 2\zeta^3 + 19\zeta^2 + \zeta + 18 & -6\zeta^5 - 12\zeta^4 - 4\zeta^3 - 10\zeta^2 - 2\zeta - 22 & -2\zeta^5 + 10\zeta^4 + 8\zeta^3 - 8\zeta^2 - \end{array}\right.$$

respectively.