The Boundary Action of the Basilica Group

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The Basilica Group

The Basilica group B is a group of automorphisms of the rooted binary tree

∘) $X = \{0, 1\}$

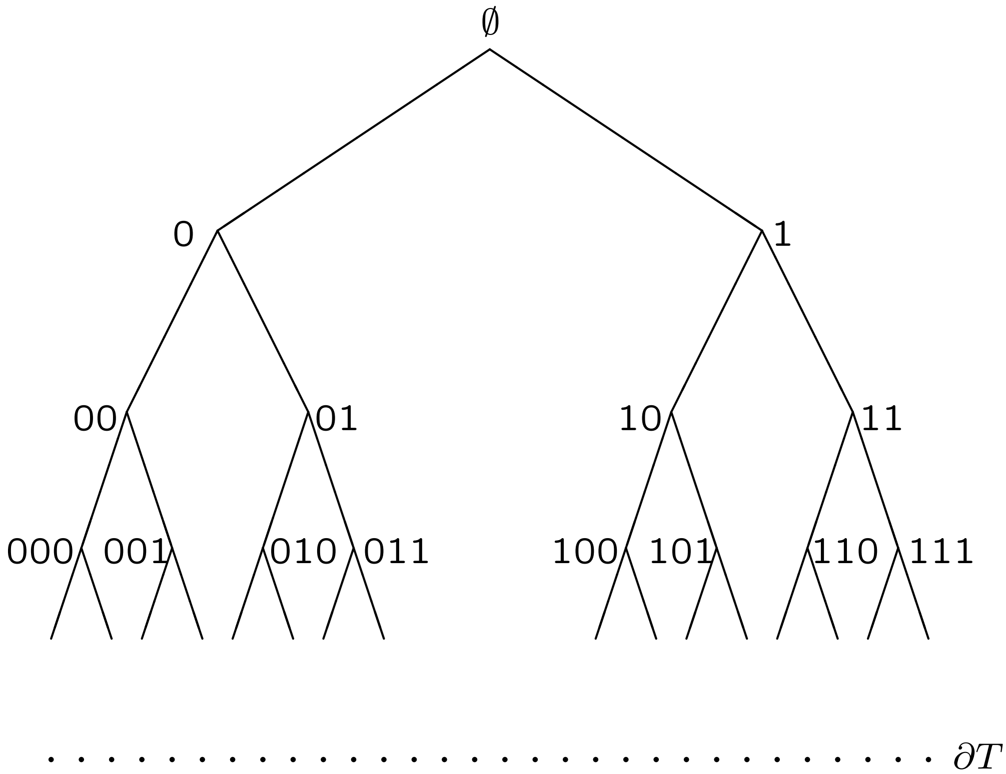
∘) $T =$ rooted binary tree, $VT = X^*$;

∘) $B < Aut(T)$;

∘) B is a self-similar group generated by the automorphisms

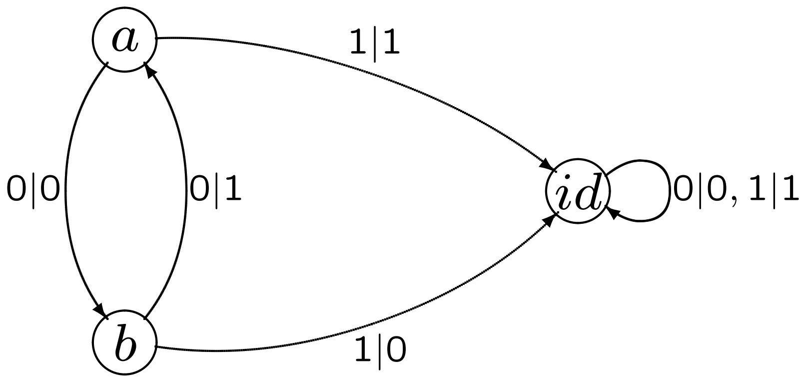
$$a = (b, id)e \qquad b = (a, id)\epsilon$$

∘) ∂T = space of ends = infinite words in $\{0, 1\} = X^\omega$.



The rooted binary tree

It was introduced by R. Grigorchuk and A. Żuk as the group generated by the following 3-state automaton.



$$a = (b, id)e, \;\; b = (a, id)\epsilon$$

Some properties:

$\diamond$) B is not *subexponentially amenable* (Grigorchuk-Żuk) but it is *amenable* (Bartholdi-Virag) (see also Kaimanovich Münchhausen Trick);

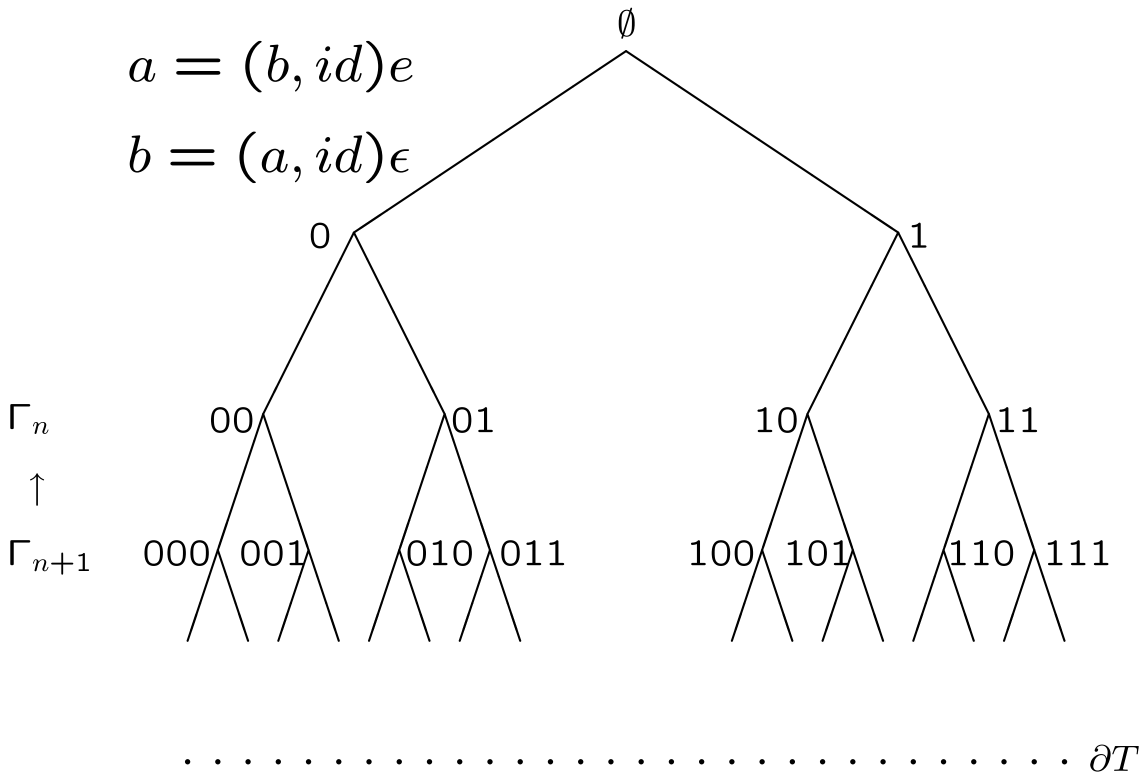
$\diamond$) B is the *Iterated Monodromy Group* of the polynomial $z^2 - 1$ (Nekrashevych).

Basilica Schreier graphs

Each vertex of the n-th level of T is identified with an element of $\{0, 1\}^n$; the boundary of T is identified with $\{0, 1\}^\omega$.

$$a = (b, id)e$$

$$b = (a, id)\epsilon$$

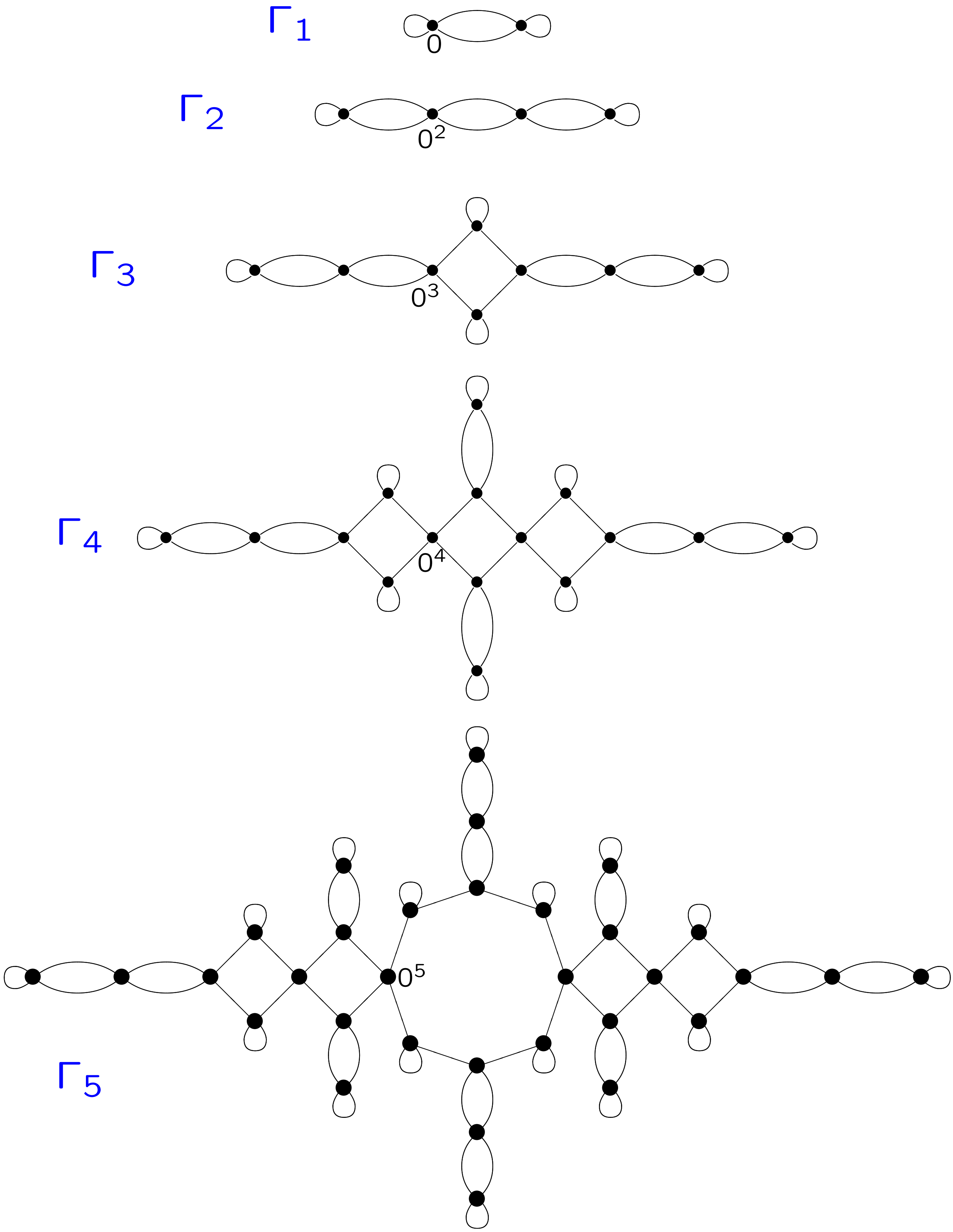


Γ_n is the Schreier graph of B w.r.t. the subgroup $Stab_B(w)$, with $|w| = n$

Remark The map $\pi_{n+1} : \Gamma_{n+1} \longrightarrow \Gamma_n$ defined as

$$\pi_{n+1}(x_1 \ldots x_n x_{n+1}) = x_1 \ldots x_n$$

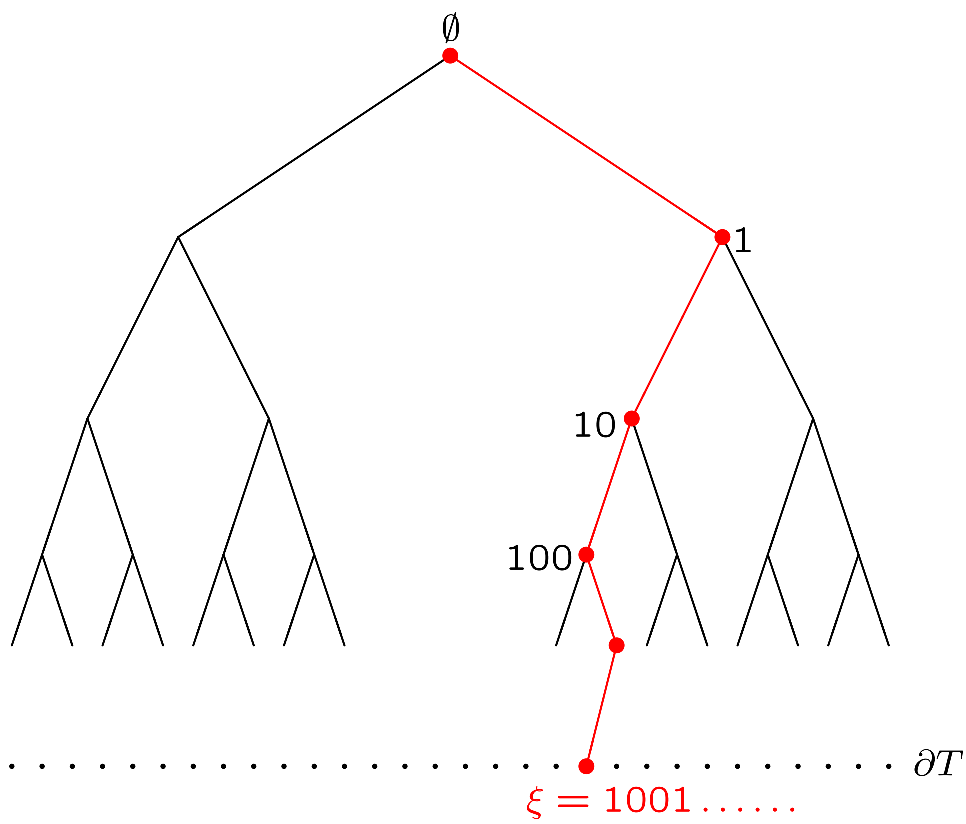
is a graph covering of degree 2.


Γ_1
0
Γ_2
0^2
Γ_3
0^3
Γ_4
0^4
0^5
Γ_5

Convergence in the space of rooted graphs

Definition The sequence of rooted graphs (Σ_n, o_n) converges (in the sense of Gromov-Hausdorff) to the graph (Σ, o) if for all $R > 0$ there exists N s.t. for every $n > N$ one has $B_{\Sigma_n}(o_n, R) = B_{\Sigma}(o, R)$.

Let $\xi = x_1 x_2 \ldots \in \partial T$, we set $\xi_n = x_1 \ldots x_n$.
- $B \curvearrowright \partial T \to$ orbital Schreier graphs Γ_ξ.
We have $(\Gamma_n, \xi_n) \longrightarrow (\Gamma_\xi, \xi)$.



Aim: Describe all limits of finite Schreier graphs in this topology. In other words classify infinite Schreier graphs Γ_ξ, $\xi \in \partial T$.

Motivations: Limit graphs of $\{\Gamma_n\}$ can be studied from different points of view:

- *spectral analysis* (Grigorchuk, Bartholdi, Nekrashevych, Teplyaev, Šunić et al.)
- *asymptotic tree entropy* (R. Lyons)
- *dimer model* (D-Donno-Nagnibeda, Teufl-Wagner)
- *Ising model* (D-Donno-Nagnibeda)
- *sandpile model* (Matter-Nagnibeda)

Also more generally: spectral analysis + probability + statistical physics on fractals (Barlow, Teplyaev, Lapidus, Sabot, Kigami, Woess, Grabner et al.)

RESULTS:

- Topological classification of $\{\Gamma_\xi\}_{\xi\in\{0,1\}^\omega}$ with respect to the number of ends.
- Measurable classification.
- Classification up to isomorphism.

Let $\xi \in \partial T$ and Γ_ξ the orbit Schreier graph of the action of B on ξ. Let m be the uniform measure on ∂T.

Theorem $\forall \xi \in \partial T$, Γ_ξ has 1, 2 or 4 ends. More precisely
($\diamond$) $\partial T = E_1 \sqcup E_2 \sqcup E_4$; with

$$E_i = \{\xi \in \partial T : \Gamma_\xi \text{ has } i \text{ ends}\}$$

where $E_4 := \{w0^\omega, w(01)^\omega : \ w \text{ any finite word}\}$ and
$E_1 = \{\alpha_1\beta_1\alpha_2\beta_2 \ldots\ldots$ with $\alpha_i, \beta_i \in \{0,1\}$ s.t. both $\{\alpha_i\}_{i\geq 1}$ and $\{\beta_i\}_{i\geq 1}$ have infinitely many 1's$\}$.
($\diamond$) $m(E_1) = 1$.

Remark The action of B on ∂T consists of infinitely many orbits. E_4 is exactly one orbit, both E_2 and E_1 consist of infinitely many orbits.

Classification up to isomorphism (of un-rooted graphs)

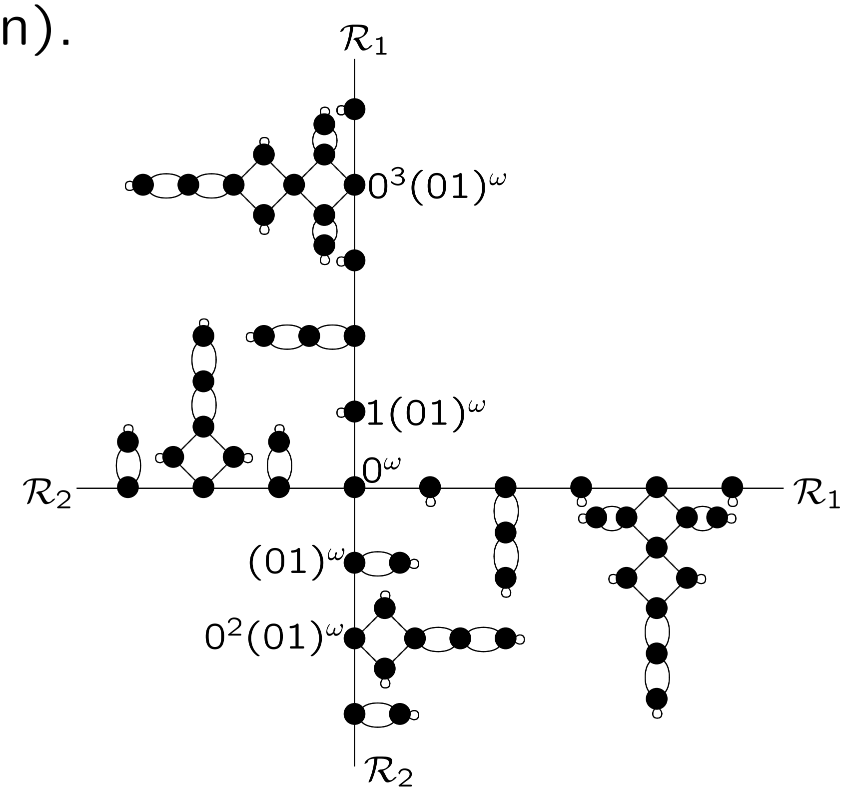
The case of 4 ends:

Corollary E_4 is an isomorphism class.

$$A_k := \left\{ n \in \mathbb{Z}; n \equiv 2^k \mod 2^{k+1} \right\}, \ k \geq 0.$$

$$\mathbb{Z} = \bigsqcup_{k \geq 0} A_k$$

$\mathcal{R}_1 = \mathcal{R}_2 = \mathbb{Z}$. Attach to each vertex $n \in A_k$ a $(2k+1)$-decoration by its unique vertex of degree 2. Do similarly for $\mathcal{R}_2$ (taking a $(2k+2)$-decoration).



The infinite Schreier graph with 4 ends

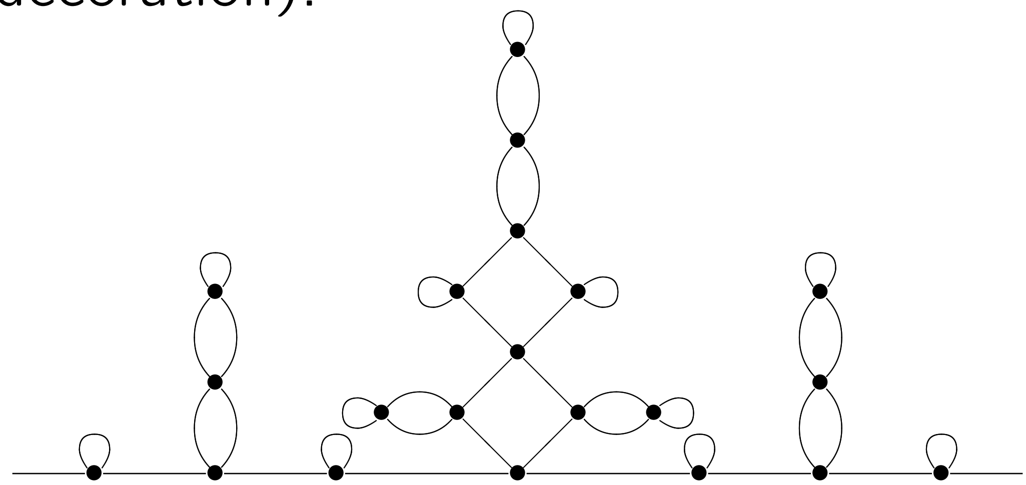
The case of 2 ends:

Theorem There exist exactly two isomorphism types of 2-ended Schreier graphs, Γ_{odd} and Γ_{even}.

The graph Γ_{odd}: take a copy of $\mathbb{Z}$ and for every $k \geq 0$ consider the partition of $\mathbb{Z}$

$$B_k := \left\{ n \in \mathbb{Z}; n \equiv \frac{1-(-2)^k}{3} \mod 2^{k+1} \right\}.$$

Attach to each vertex $n \in B_k$ a $(2k+1)$-decoration (the same for Γ_{even} taking a $(2k+2)$-decoration).



A finite part of Γ_{odd}

Let $\xi \in E_2$ such that $\xi = \alpha_1\beta_1\alpha_2\beta_2\ldots$. If ξ is such that the sequence $\{\alpha_i\}_{i\geq 1}$ (resp. $\{\beta_i\}_{i\geq 1}$) has a finite number of 1's, then the sequence $\{\Gamma_n\}$ converges to Γ_{even} (resp. Γ_{odd}).

The case of 1 end:

Theorem The set E_1 of graphs with one end contains uncountably many classes of isomorphism each one of measure 0.

Theorem Let $\xi \in E_1$. Either
(1) $\xi = u1^\omega$ for some finite words u
or
(2)
$\xi = 0^k 1 (0x_1^{(0)} \dots 0x_{m_0}^{(0)}) 1^{t_1} (0x_1^{(1)} \dots 0x_{m_1}^{(1)}) 1^{t_2} (0x_1^{(2)} \dots 0x_{m_2}^{(2)}) \dots\dots$
with $x_i^{(j)} \in \{0,1\}$
Then
- $\Gamma_\xi \simeq \Gamma_\eta$ if ξ and η are as in (1);
- $\Gamma_\xi \not\simeq \Gamma_\eta$ if ξ is as in (1) and η as in (2);
- $\Gamma_\xi \simeq \Gamma_\eta$ if ξ and η are as in (2) so that

$$\xi = 0^g 1 (0x_1^{(0)} \dots 0x_{m_0}^{(0)}) 1^{t_1} (0x_1^{(1)} \dots 0x_{m_1}^{(1)}) 1^{t_2} (0x_1^{(2)} \dots 0x_{m_2}^{(2)}) \dots\dots$$

$$\eta = 0^h 1 (0y_1^{(0)} \dots 0y_{n_0}^{(0)}) 1^{s_1} (0y_1^{(1)} \dots 0y_{n_1}^{(1)}) 1^{s_2} (0y_1^{(2)} \dots 0y_{n_2}^{(2)}) \dots\dots$$

and there exist $R, Q \geq 1$ such that, $\forall\ k \geq 0$,

$$m_{R+k} = m_{Q+k} \qquad t_{R+1+k} = t_{Q+1+k}$$

either $y_p^{(Q+k)} = x_p^{(R+k)}$ or $y_p^{(Q+k)} = 1 - x_p^{(R+k)}$, with $p = 1, \dots, m_{R+k}$

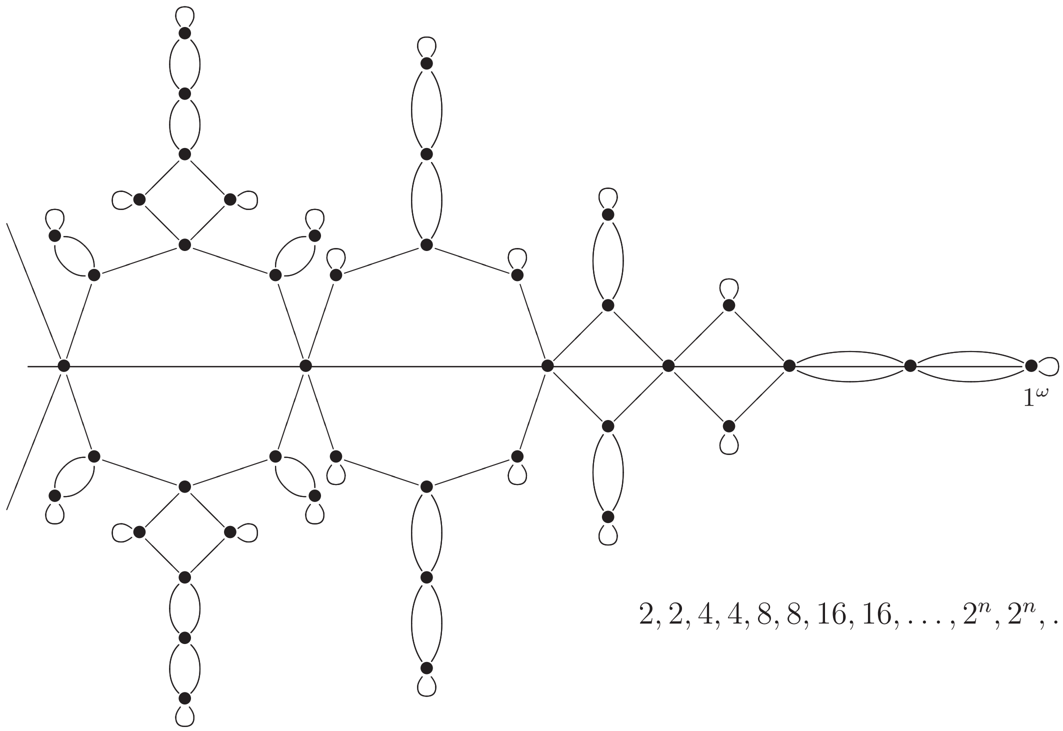
$$\text{and} \quad g + \sum_{i=1}^{R} t_i + \sum_{i=0}^{R-1} m_i = h + \sum_{k=1}^{Q} s_i + \sum_{i=0}^{Q-1} n_i$$

FINAL REMARKS

- *Heuristic correspondence with the limit space.*

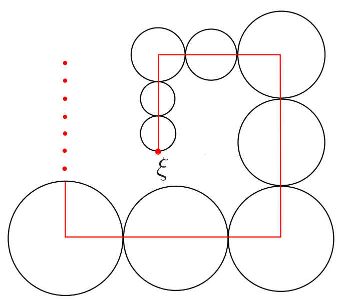
- *Similar analysis can be done for other self-similar groups, in particular for other IMG.*

- *Unimodular random graphs.*



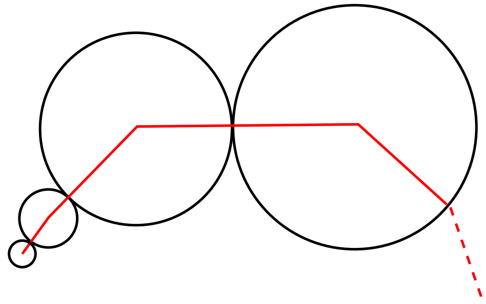
$2, 2, 4, 4, 8, 8, 16, 16, \ldots, 2^n, 2^n, \ldots$

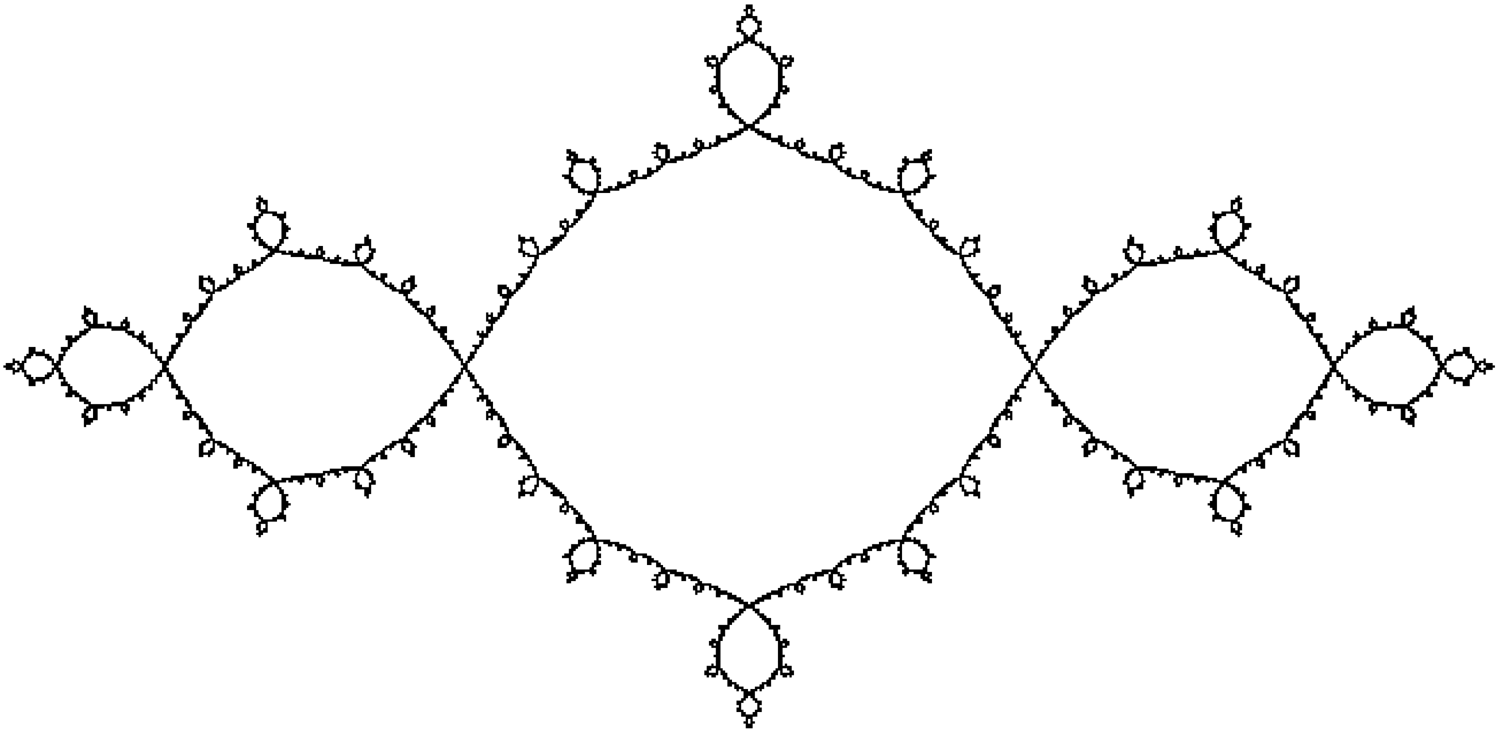
A finite part of $(\Gamma_{1^\omega}, 1^\omega)$



$\xi = 1(1100)^\omega$

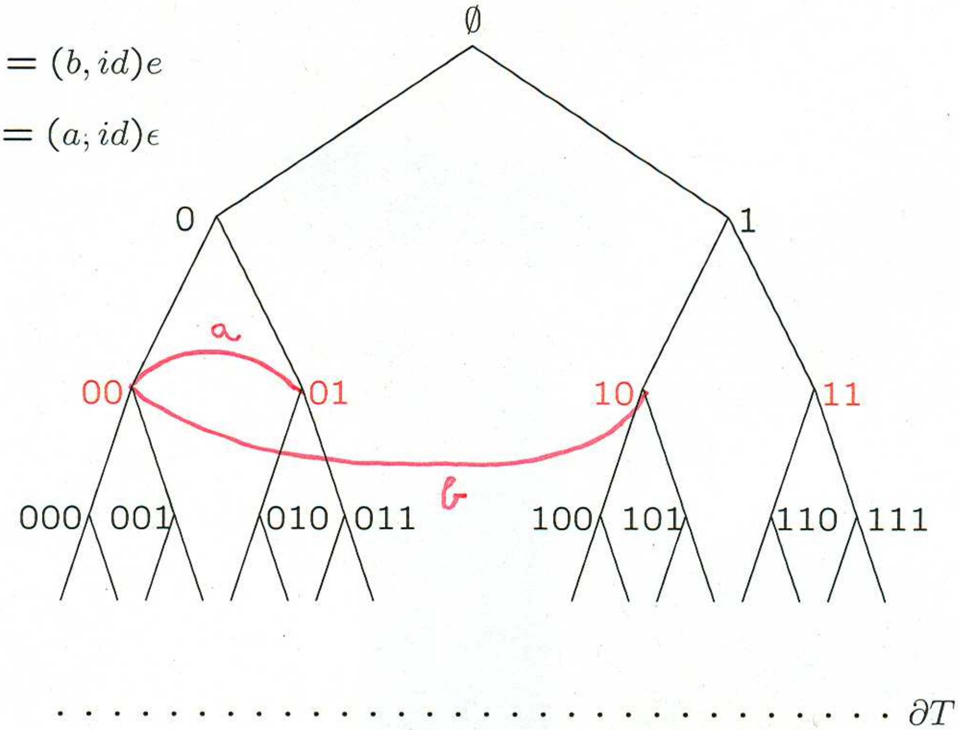
$2, 2, 8, 8, 32, 32, \ldots$





$a = (b, id)e$

$b = (a; id)\epsilon$



Γ_2