An Inaccessible Graph

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Let X be a locally finite connected graph. A *ray* is a sequence of distinct vertices $v_0, v_1, \ldots$ such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots$. For a ray to exist the graph X has to be infinite.

We say that two rays R, R' belong to the same *end* ω if for no finite subset F of VX or EX do R_1 and R_2 eventually lie in distinct components of $X \setminus F$. We say that ω is *thin* if it does not contain infinitely many vertex disjoint rays Thomassen and Woess (1993) define an end ω to be *thick* if it is not thin.

Thomassen and Woess also define an accessible graph. A graph X is *accessible* if there is some natural number k such that for any two ends ω_1 and ω_2 of X, there is a set E of at most k edges in X such that E separates ω_1 and ω_2 i.e. removing E from X disconnects the graph in such a way that rays R_1, R_2 of ω_1, ω_2 respectively eventually lie in distinct components of $X \setminus E$.

A finitely generated group G is said to have more than one end ($e(G) > 1$) if its Cayley graph $X(G, S)$ with respect to a finite generating set S has more than one end. This property is independent of the generating set S chosen. Stallings (1971) showed that if $e(G) > 1$ then G *splits* over a finite subgroup, i.e. either $G = A *_C B$ where B is finite $C \neq A, C \neq B$ or G is an HNN extension $G = A*_C = \langle A, t | t^{-1}ct = \theta(c)\rangle$, where $C \leq A$ and $\theta : C \to A$ is an injective homomorphism.

A group is *accessible* if the process of successively factorizing factors that split in a decomposition of G eventually terminates with factors that are finite or one ended. Thomassen and Woess show that the Cayley graph of a finitely generated group G is accessible if and only if G is accessible. In a number of papers I have given examples of inaccessible groups and so not every locally finite connected graph is accessible.

Let ω be an end of X. Following Thomassen and Woess define $k(\omega)$ to be the smallest integer k such that ω can be separated from any other end by at most k vertuces. If this number does not exist put $k(\omega) = \infty$.

Thomassen and Woess show that X is accessible if and only if $k(\omega) < \infty$ for every end ω. We say that an end ω is *special* if $k(\omega) = \infty$.

I will describe a locally finite, connected, inaccessible, vertex transitive graph X which is not quasi-isometric to a Cayley graph. The properties of being inaccessible is invariant under quasi-isometry.

Woess asked in 1989 if every vertex transitive locally finite graph is quasi-isometric to a Cayley graph. It was shown in Eskin, Fisher and Whyte in 2007 that the Diestel-Leader graph $DL(m,n), m \neq n$ is not quasi-isometric to a Cayley graph, answering the question of Woess. The graph described here is another example.

In 1998 Mary Jones and I constructed a finitely generated group G for which $G \cong A *_C G$ where C is infinite cyclic. The vertex set of the graph X is the set of left cosets of A in G. The construction of G is as follows.

The graph

Let

$$A = \langle a, b | b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$

Then a^2 is in the centre of A and $A/\langle a^2 \rangle \cong S_3$. Also A is generated by a^3 and a^2b since

$$a^{-3}(a^2b)a^3 = a^2b^{-1},$$

and so

$$b^2 = b^{-1} \in \langle a^3, a^2b \rangle.$$

The group A has a lattice of subgroups as in Fig 1.

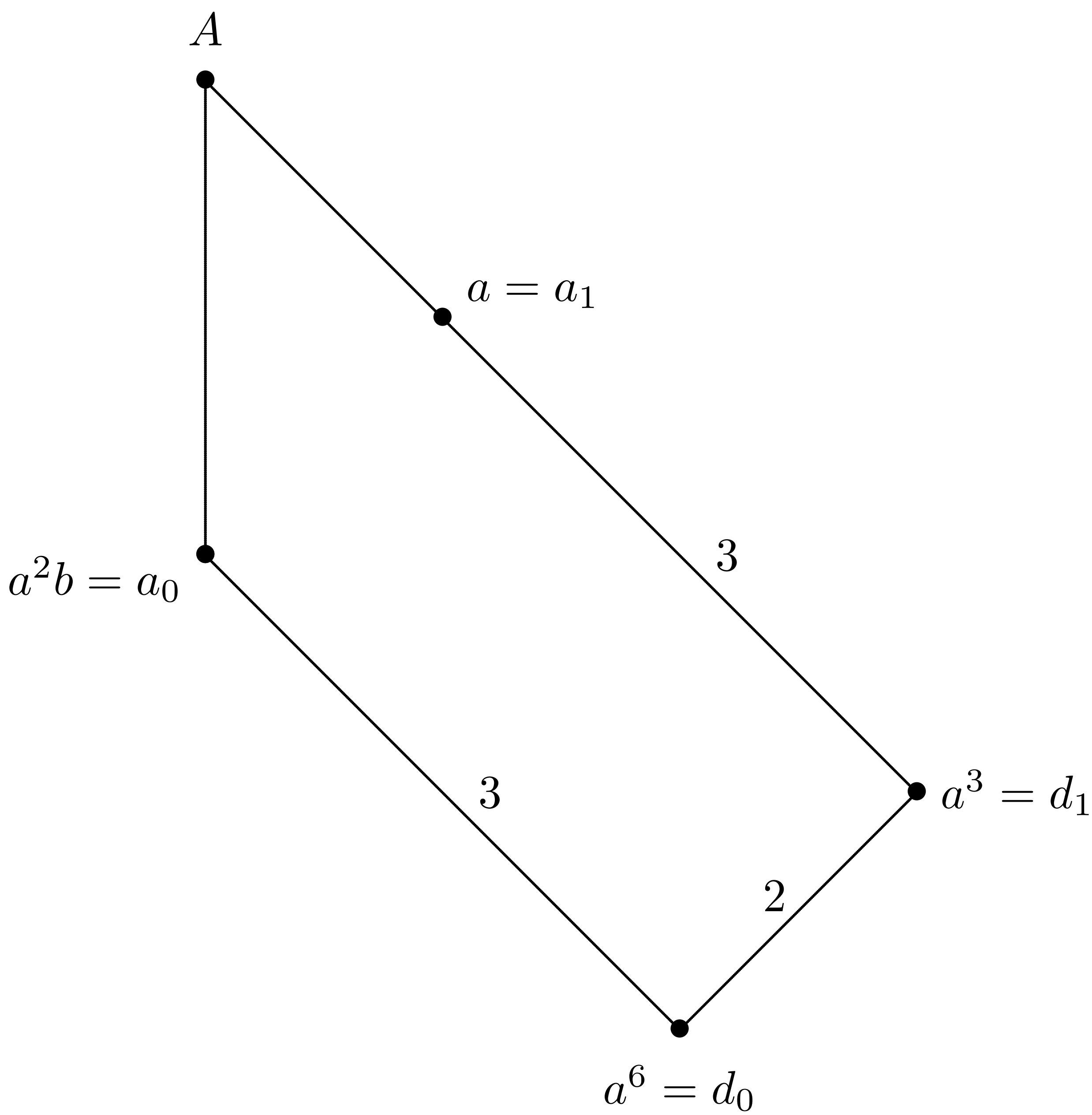

Fig 1

The group G is generated by four elements a, b, c and d, subject to an infinite set of defining relations as follows. The elements a, b satisfy the relations of A

and in fact generate a subgroup of G isomorphic to A.

Then $c^{-1}dc = d^2$, so that c, d generate a subgroup isomorphic to the Baumslag-Solitar group $BS(1,2)$. Also $a^3 = d$ is another relation. The remaining relations are defined inductively. Put $d = d_1, a = a_1$ and $d_{i+1} = cd_ic^{-1}$ so that $d_{i+1}^2 = d_i$.

Put $d_0 = d_1^2$ and $a_0 = a^2b$. Then as above the subgroup $A = \langle a, b\rangle = \langle a_0, d_1\rangle$. Now define inductively $a_{i+1} = a_id_{i+1}^{-1}a_i, b_{i+1} = a_i^{-1}d_{i+1}a_id_{i+1}^{-1}$ and add the relations $b_{i+1}^3 = 1, a_{i+1}^{-1}b_{i+1}a_{i+1} = b_{i+1}^{-1}$ for each i to make $A_{i+1} = \langle a_{i+1}, b_{i+1}\rangle \cong A$. Note that for $i = 1$ we have $a = a_1 = a_0d_1^{-1}a_0 = yx^{-1}y$ as above. The group G is best understood in terms of the subgroup lattice shown in Fig 2.

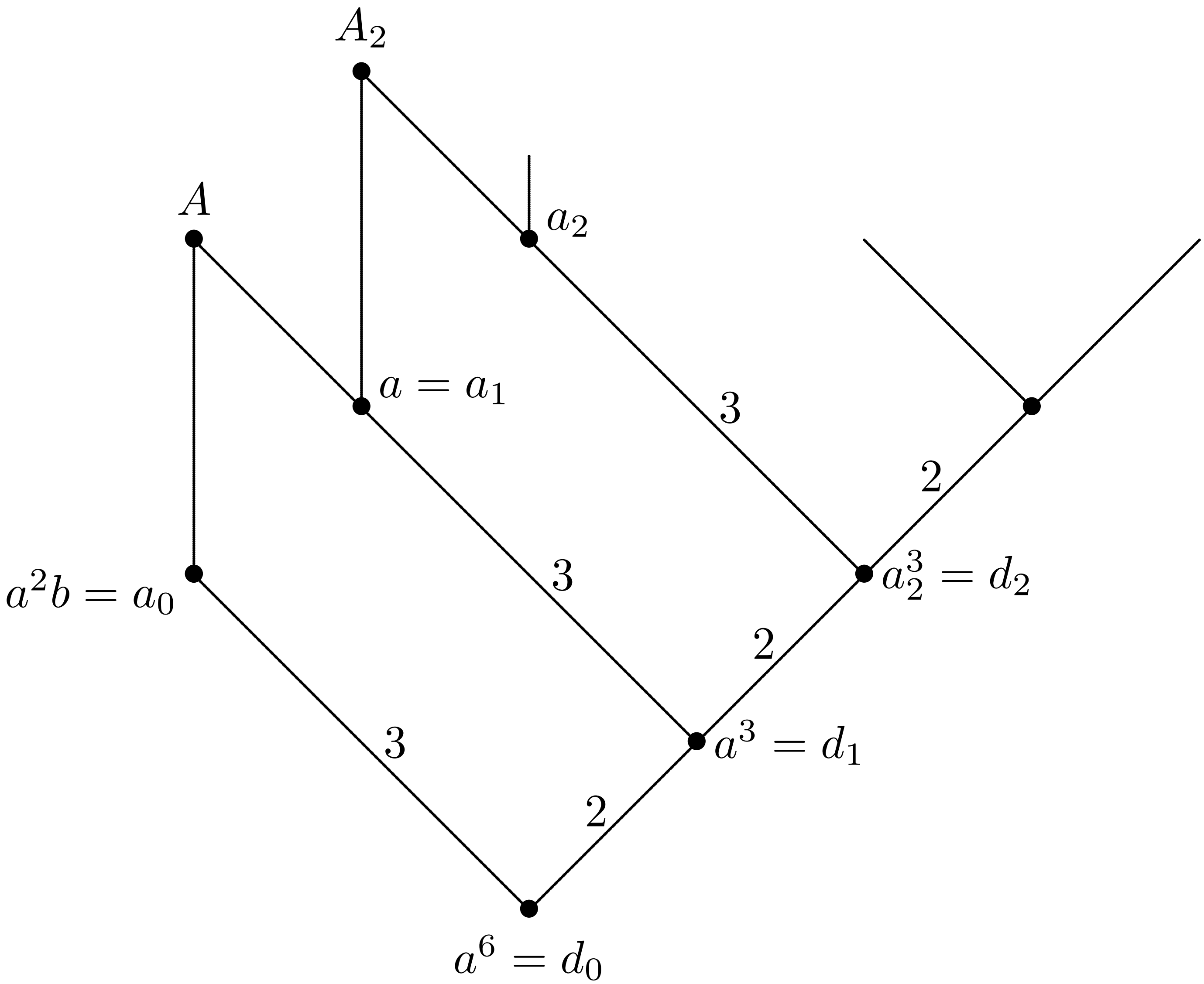

Fig 2

It can be seen that $G \cong A *_C G$ where $A = \langle a, b \rangle = \langle a_0, d_1 \rangle$ and $C = \langle a_1 \rangle$.

Let $D = \langle d_1 \rangle$. Let Y be the G-graph with two orbits of vertices $VY = \{gA, gD | g \in G\}$ and two orbits of edges $EY = \{(gA, gD), (gD, gcD) | g \in G\}$.

In Y the vertex A is incident with $[A, D] = 9$ edges, as is every vertex in the same orbit. The vertex D is incident with 4 edges. One edge in one edge orbit connects D to A and there are three edges in the other orbit connecting D to $cD, c^{-1}D$ and $dc^{-1}D$. Note that $d = d_1$ fixes the edge (D, cD) and transposes the edges $(D, c^{-1}D), (D, dc^{-1}D)$. If one removes the edges of Y in the first orbit one is left with a set of 3-regular trees. If one directs these subgraphs by putting an arrow from D to cD then every vertex has one edge pointing away from it and two pointing towards it as in Fig 3.

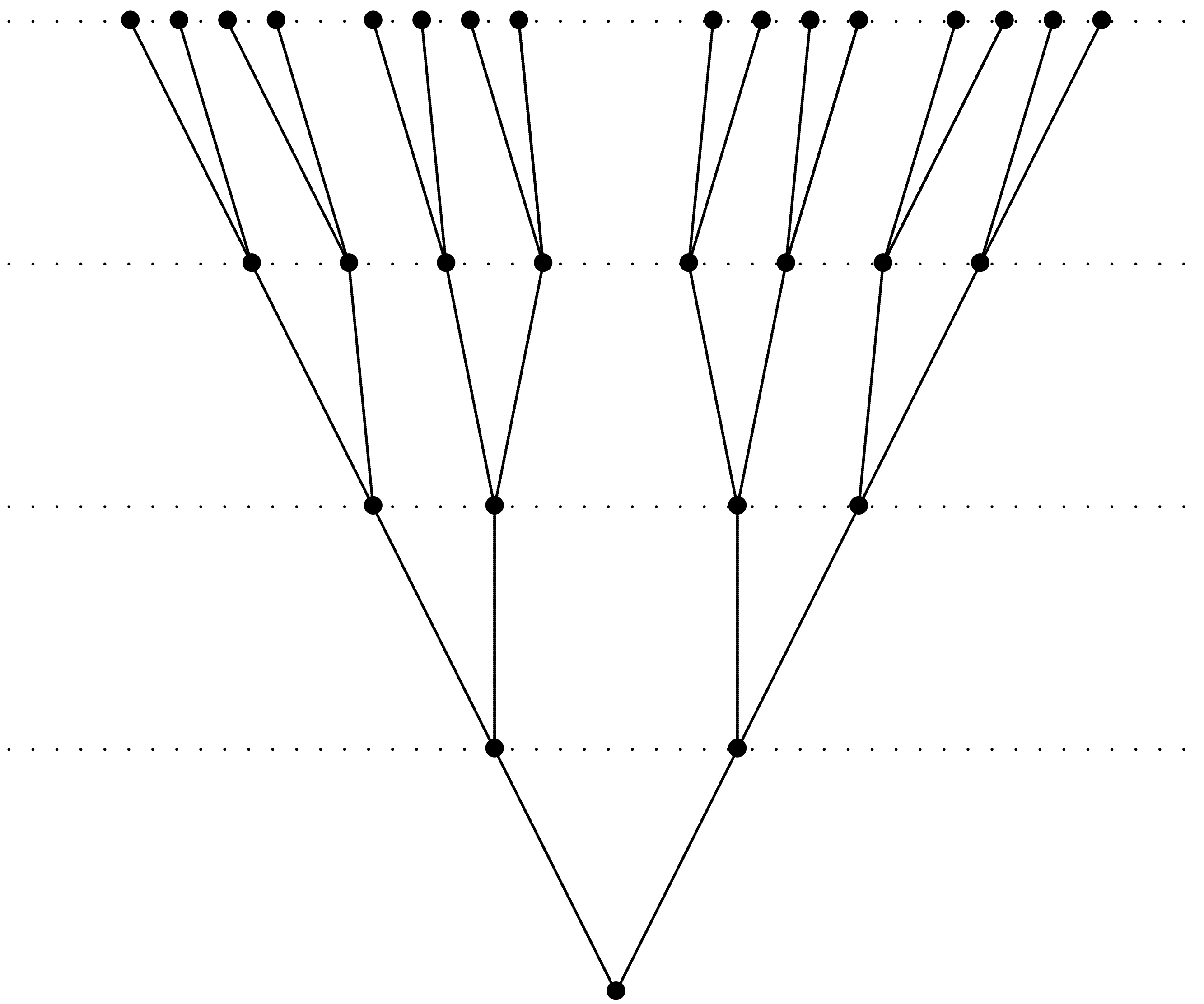

Fig 3

The graph Y is connected because G is generated by A, D and c. One obtains a vertex transitive G-graph X from Y by taking one orbit of vertices and joining two vertices by an edge if they are distance two apart in Y. If the orbit chosen contains D then each vertex will have degree 10. It is easier to work with the graph Y which is quasi-isometric to X.

In fact in Y each of the A vertices is a cut point. There is a structure tree decomposition of Y, $VY = C \cup B$ where C is the set of cut points (in this case the vertices in the orbit of A) and B is the set of 2-blocks (maximal 2-connected subgraphs). In this case each 2-block is quasi-isometrc to the original graph Y. The quasi-isometry can be achieved by a pair of *folding* operations.

Removing the vertex A from Y results in 3 components. In each 2-block a vertex in the orbit of A has degree 3.

It is now indicated why Y is inaccessible and not quasi-isometric to a Cayley graph.

Let Z be a subgraph of Y that is a 3-regular tree. There are countably many such subgraphs. Any two rays in Z are in the same end. This is because any two vertices of Y that are at the same level (indicated by a dotted line in Fig 3) are joined by a path in Y

that contains no other edge of Z. Thus there is an end corresponding to Z. This end ω is special $(k(\omega = \infty$. Not every special end of Y corresponds to a subgraph like Z. However there are only countably many special ends. An end belonging to a subgraph like Z is called *very special*.

Suppose Y is quasi-isometric to a Cayley graph W. Then W will be inaccessible with countably many special ends. A quasi-isometry between Y and W will take a very special end of Y to a very special end of W, i.e. an end corresponding to a subgraph which is quasi-isometric to a regular tree.

Inaccessible Cayley graphs with countably many special ends do exist. However in such a Cayley graph if an end is determined by rays in a subgraph, that subgraph would have to be one-ended. We have seen that the 3-regular tree Z has infinitely many ends.

I do not think that Y has any one-ended locally finite subgraphs.