Heat transport to the boundary on discrete graphs

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Workshop on „Boundaries“ Graz, 01.07.2009

Plan

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2. Stochastic completeness for the bounded case

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1. Introduction

Let $G = (V, E)$ be a connected graph.

Let Δ on $\ell^2(V)$ be given by

$$(\Delta\varphi)(x) = \sum_{y\sim x} \varphi(x) - \varphi(y).$$

Let $\widetilde{\Delta}$ be the formal extension of Δ to all $\varphi : V \to R$ such that

$$\sum_{y\sim x} \varphi(x) - \varphi(y) < \infty.$$

The heat equation

The heat equation (HE)

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$$\begin{aligned} -\widetilde{\Delta} u_t(x) &= \partial_t u_t(x) \\ u_0(x) &= f(x) \end{aligned}$$

is solved by

$$u_t(x) = e^{-t\Delta} f(x),$$

In particular it is the distribution of heat at time $t \geq 0$ for initial distribution f.

Question

Which amount of heat is in the graph at time $t > 0$,
if it was distributed as $f \in \ell^1(V)$ at time $t = 0$?

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$$\begin{aligned}
\sum_{x \in V} e^{-t\Delta} f(x) &= \langle 1, e^{-t\Delta} f \rangle \\
&= \langle e^{-t\Delta} 1, f \rangle \\
&= \sum_{x \in V} f(x) \underbrace{e^{-t\Delta} 1(x)}_{0 \leq \ldots \leq 1}.
\end{aligned}$$

Stochastic (in-)completeness

The operator Δ (the graph G) is called **stochastically complete**.

$:\Longleftrightarrow\ e^{-t\Delta}1 = 1$ for all $t > 0$.

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$:\Longleftrightarrow \; e^{-t\Delta}1(x) < 1$ for some $t > 0$ and $x \in V$.

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2. Stochastic completeness for the bounded case

Let $\deg(x)$ be the degree of a vertex $x \in V$.

Theorem. (Dodziuk, Mathai '06)
If $\sup_{x\in V} \deg(x) < \infty$. Then

$$(\Delta\varphi)(x) = \sum_{y\sim x} \varphi(x) - \varphi(y), \quad \text{on } \ell^2(V)$$

is stochastically complete, i.e. $e^{-t\Delta}1 = 1$ for all $t > 0$.

More on bounded operators

Fact: Δ bounded $\Leftrightarrow \sup_x \deg(x) < \infty$.

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In particular

$$I - P : \ell^2(V, \deg) \to \ell^2(V, \deg)$$

$$((I-P)\varphi)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} \varphi(x) - \varphi(y)$$

is stochastically complete on every graph since it is always bounded.

What happens if $\sup_x \deg(x) = \infty$?

3. Stochastic incompleteness and the ℓ^∞ spectrum

- What is the operator?
  - Wojciechowski '07
  - Weber '08
  - Jorgensen '08 (weighted graphs)

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- Stochastic completeness
  - Sturm '94 (strongly local Dirichlet forms)
  - Grigor'yan '99 (Riemannian manifolds)
  - Wojciechowski '07 (Δ on graphs)
  - Weber '08

Dirichlet forms

Let V be countable and $m : V \to (0, \infty)$ a measure.

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All **regular Dirichlet forms** $Q : \ell^2(V, m) \to [0, \infty]$ are given by

$$Q(\varphi) = \frac{1}{2} \sum_{x,y \in V} b(x, y)(\varphi(x) - \varphi(y))^2 + \sum_{x \in V} c(x)\varphi(x)^2$$

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with $c : V \to [0, \infty)$ and $b : V \times V \to [0, \infty)$:

- $b(x,y) = b(y,x)$
- $\sum_y b(x,y) < \infty \text{ all } x \in V$.

What is the operator?

By polarization and general theory

$$Q(\varphi, \psi) = \langle L^{\frac{1}{2}}\varphi, L^{\frac{1}{2}}\psi\rangle,$$

where L on $D(L) \subseteq \ell^2(V, m)$ is given by

$$(L\varphi)(x) = \frac{1}{m(x)} \sum_{y \in V} b(x,y)(\varphi(x) - \varphi(y)) + \frac{c(x)}{m(x)}\varphi(x).$$

Let $\widetilde{L}$ be the formal extension of L to all φ such that RHS$< \infty$.

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For $t \geq 0$ let $M_t : V \to [0, \infty)$

$$M_t \;=\; \underbrace{e^{-tL}1}_{\text{heat in the graph}} + \underbrace{\int_0^t e^{-sL} c \, ds}_{\text{heat killed by } c} .$$

Question: Is $M_t < 1$?

When is $M_t < 1$?

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- Sturm '94 (strongly local Dirichlet forms)
- Grigor'yan '99 (Riemannian manifolds)
- Wojciechowski '07 (Δ on graphs)

Examples

Theorem. (Wojciechowski '07)
If for every non-repeating path (x_n)

$$\sum_{n\in\mathbb{N}} \frac{1}{\deg_+(x_n)} < \infty.$$

Then

$$e^{-t\Delta}1 < 1.$$

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Question: Does existence of a stochastically incomplete subgraph imply stochastic incompleteness of the graph?

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No!

Theorem. (K., Lenz'09)
Every graph is a subgraph of a stochastically complete graph.

Which additional assumptions are sufficient?

Theorem. (K., Lenz'09)
If the operator with Dirichlet boundary conditions on a subgraph is stochastically incomplete, then the operator on the graph is stochastically incomplete.

5. Outlook and open questions?

(1) Given an operator L.
Is there a metric d_L such that for the distance balls $B_r(x)$ the following holds: If

$$\sum \frac{r}{\log |B_r(x)|} = \infty,$$

then L is stochastically complete?

5. Outlook and open questions?

(2) Given an operator L.
Can one find

- a boundary ∂V of a graph,
- a metric d'_L which extends to ∂V,
- a measure m on the boundary ∂V.

such that if for some $x \in V$

$$m(\{y \in \partial V \mid d'_L(x,y) < \infty\})$$

is 'large', then L is stochastically incomplete.